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Construction of Galois Fields of Characteristic Two and Irreducible Polynomials

By J. D. Swift

1. Introduction. The primary purpose of this paper is to provide a practical method of constructing Galois fields of characteristic 2 of large orders and thereby simultaneously give a practical way of generating a large supply of polynomials of high degree irreducible modulo 2. These polynomials are of special interest in connection with the theory of linear recursive sequences. See, for example [5]; some applications are suggested in [4].

The basic structure of Galois fields is extremely simple. For each prime q and each n there is one and (up to isomorphism) only one finite field of order q^n, designated by $GF(q^n)$. Its additive group is the elementary abelian group; the direct sum of n cyclic groups of order q. The multiplicative group of the non-zero elements is cyclic. The field $GF(q^{nm})$ may be constructed from a given $GF(q^n)$ by finding a polynomial of degree m, irreducible over $GF(q^n)$ and considering the set of polynomials with coefficients in $GF(q^n)$ modulis q and the irreducible polynomial. The subfields of $GF(q^n)$ are precisely $GF(q^d)$ where $d \mid n$; if g is a primitive generator of $GF(q^n)$, i.e. of the multiplicative subgroup of its non-zero elements, g^m is a primitive generator of $GF(q^d)$ where $m = (q^n - 1)/(q^d - 1)$.

These theorems, covered by many basic texts of algebra since the original work of L. E. Dickson [2], essentially dispose of the elementary theory of these fields. A particularly complete theory is contained in [1] which also has a bibliography listing a number of items from the extensive literature of the deeper arithmetic of the Galois fields. In general, results quoted without proof in this paper may either be found directly in [1] or are immediate consequences of statements proved therein.

From a practical standpoint, the only problems left by the structure theorems are those of finding an irreducible polynomial of degree m over the base field and of finding a primitive generator of the field with respect to this polynomial. In certain cases an irreducible polynomial is readily available and we are here concerned with the exploitation of these cases.

2. Cyclotomic polynomials over $GF(2)$. Let p be a prime for which 2 is a primitive root; then the cyclotomic polynomial $f_p(x) = (x^p + 1)/(x + 1)$ is irreducible over $GF(2)$. Thus for such primes the theory permits the realization of $GF(2^{p-1})$. Further, if g is a primitive generator of this field we may realize $GF(2^d)$ by considering powers of g^m, $m = (2^{p-1} - 1)/(2^d - 1)$ when $d \mid (p - 1)$. Since the only obvious restriction imposed on p by the condition that 2 be a primitive root of p is that $p \equiv \pm 3 \pmod 8$, it is likely that all fields $GF(2^n)$, $n \not\equiv 0 \pmod 8$, may be realized in this way for sufficiently large p. The smallest fields which cannot be constructed from polynomials listed in this paper are those for which $n = 8, 16, 17$.

If z is any element of $GF(2^n)$ the powers $z^0, z^1, \cdots, z^n$ will be linearly dependent over $GF(2)$ and the resulting relation of dependence $f(z) = 0$ will give an irreducible

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polynomial $f(z)$ if z does not lie in a proper subfield of $GF(2^n)$. If z is a primitive generator of $GF(2^n)$, $f(z)$ will be, by definition, primitive irreducible. All irreducible polynomials over $GF(2)$ may be constructed in this way.

By simple counting arguments we see that the number of irreducible polynomials of degree n is

$$\frac{1}{n}\left(2^n - \Sigma 2^{n/q_i} + \Sigma 2^{n/q_i q_j} - \cdots\right)$$

where the q_i are the distinct prime divisors of n. Similarly, the number of primitive irreducible polynomials of this degree is $(1/n)\varphi(2^n - 1)$ where φ is the Euler totient.

The actual problem of construction of these fields now reduces to finding a generator of the cyclic group. It is at this point that the theory carries us no further and resort must be taken to high speed computing machinery. It is easy to see that, for $p > 5$, any generator modulo the cyclotomic polynomial must be of degree at least 3. In particular, since $x^p = 1$, x and x^2 are of order p. To investigate the orders of the other linear and quadratic forms we study $z = x + 1/x$; define $h(z) = x^{-m} f_p(x)$ where $m = (p - 1)/2$; h is a polynomial of degree m in z. Thus z belongs to the sub-field $GF(2^m)$ and the order of z is a divisor of $2^m - 1$. Now $(x + 1)^2 = x^2 + 1 = xz$; thus the order of $x^2 + 1$ is a divisor of $p(2^m - 1)$ and the order of $x + 1$ is the same as that of its square. Again $x(x + 1)$ has the same order as $x + 1$. Finally $x^2 + x + 1 = x(z + 1)$ and $z + 1$ is also in $GF(2^m)$. Thus the order of all linear and quadratic forms divides $p(2^m - 1) < 2^{p-1} - 1$ for $p > 5$.

It is of interest to note that the order of z is precisely $2^m - 1$ for all suitable primes $p \leqq 139$ except for 37 and 101 for which it is $\frac{1}{3}$ the maximum.

We must, then, seek among the cubic or higher polynomials for our generators and we turn to a consideration of the computations by which this may be done.

3. The power routine. The primary tool in the investigation is a high-speed routine programmed for SWAC which finds prescribed powers of polynomials $F(x)$ modulis polynomials $G(x)$ and 2. The routine has two parts and the second part may be used as many times as desired without return to the first. The initial part receives $F(x)$ and $G(x)$ as inputs and computes n successive squares $(F(x))^2$, $(F(x))^4, \cdots, (F(x))^{2^n}$ where n is the degree of $G(x)$, reducing the results modulis 2 and $G(x)$. The last square is a check; if $G(x)$ is irreducible, $(F(x))^{2^n} = F(x)$. The second receives as input the exponent of the power of $F(x)$ desired, the upper limit being $2^n - 1$, and computes this power by multiplying together the appropriate stored powers from the first routine. The routines are quadruple precision and, since SWAC has a 36 bit word, degrees and powers are limited to 143 and $2^{143} - 1$ respectively. About $\frac{1}{2}$ second per multiplication is required in either routine. This means a maximum of 75 seconds for completion in either case.

To apply this routine to the problem of determining a primitive generator, $f_p(x)$ is used for $G(x)$ and possible generators are used as $F(x)$ in lexicographic succession. The exponents are $(2^{p-1} - 1)/q_i$ where q_i are the prime divisors of $(2^{p-1} - 1)$. (It is a fortunate circumstance that all necessary factorizations are known [3].) If the power of $F(x)$ is 1 for any of these exponents, $F(x)$ is not a primitive generator, and conversely. As a check the power is also calculated for $q_i = 1$; here the value must be 1. Table 1 lists the earliest primitive generator for each prime.

TABLE 1

Generating polynomials for $GF(2^{p-1})$*; to be used modulis* 2 *and* $(x^p + 1)/(x + 1)$

p	polynomial
3	x
5	$x + 1$
11	$x^3 + x + 1$
13	$x^3 + x + 1$
19	$x^4 + x + 1$
29	$x^5 + x^3 + x + 1$
37	$x^5 + x^2 + 1$
53	$x^4 + x + 1$
59	$x^3 + x + 1$
61	$x^3 + x + 1$
67	$x^5 + x^2 + 1$
83	$x^3 + x + 1$
101	$x^5 + x^4 + x^2 + x + 1$
107	$x^3 + x + 1$
131	$x^3 + x + 1$
139	$x^4 + x + 1$

4. Generators of subfields of the cyclotomic fields. There are 16 primes covered by Table 1. If subfields are considered, 24 additional fields may be constructed. Specifically, for each polynomial in Table 1 as $F(x)$ and its corresponding $f_p(x)$ as $G(x)$ we compute the $(2^{p-1} - 1)/(2^d - 1)$-th power of $F(x)$ for each divisor d of $p - 1$, $2 < d < p - 1$. The results are tabulated in Table 2. Here, in the largest case, the result is a polynomial of degree 138 and we have adopted a condensation of coefficients into octal notation to bring the result within manageable proportions. For example, $x^7 + x^5 + x^4 + x^2 + x + 1$ may be represented first by the ordered octuple of its coefficients: 10 110 111, and these may be read in groups of three as octal numbers. Thus the polynomial would appear as 267.

The polynomials of Table 2 are not necessarily the lexicographically earliest generators of their fields. It would be quite impossible to find such generators. However the disadvantages of polynomials of large degree are not as great as might be assumed at first thought. If a number of powers are to be computed, the full size of the registers is needed for the reduction modulo $f_p(x)$ and no more additions are needed to compute the irreducible polynomials than if the degrees were smaller. The only unavoidable disadvantage is the danger of an error in transcription. The entries in the table have been preserved against error by a comparison of output decks on successive runs and careful proofreading.

5. Utilization of generators to produce irreducible polynomials. As indicated in section 2, an irreducible polynomial is produced from an element z of $GF(2^n)$ not belonging to a proper subfield by finding the relation of dependence of $z^0, z^1, z^2, \cdots, z^n$. In one set of $n + 1$ registers are initially stored the powers of z. Another companion set is loaded initially with a single 1 in the ith place, $i = 0(1)n$. Whenever an operation is performed in one set of registers, the same is done to the companion set. By addition of other elements of the power registers the value 0 is obtained in the z^0 (or z^n) register. The irreducible polynomial is then read from

TABLE 2

Generating polynomials for proper sub-fields of $GF(2^{p-1})$. Fields are $GF(2^d)$ for $d \mid p-1$, $d \geq 3$. Generators are to be used modulis 2 and $(x^p+1)/(x+1)$. See section 4 for use of octal notation to represent polynomials.

p	d	polynomial									
11	5										360
13	6										1760
	4										1156
	3										1321
19	9									4	77744
	6									1	26202
	3									7	63175
29	14								10777	77705	
	7								6633	66330	
	4								17577	56617	
37	18							64	00000	00055	
	12							75	23334	20271	
	9							30	01030	20031	
	6							71	22347	11234	
	4							76	45677	73746	
	3							75	23377	31274	
53	26						117	77777	77777	77744	
	13						117	42534	36165	21744	
	4						26	50100	21003	22003	
59	29						3777	77777	77777	77760	
61	30						17777	77777	77777	77760	
	20						17723	64723	51515	17760	
	15						17003	60103	02017	00360	
	12						34564	50135	07663	71022	
	10						55263	06377	77143	15265	
	6						14477	11710	23623	74461	
	5						76200	51450	24624	01174	
	4						36544	52035	07653	31207	
	3						10124	50147	46024	52020	
67	33					64	00000	00000	00000	00055	
	22					75	24504	50000	00452	35255	
	11					2	13043	36044	17304	32101	
	6					25	46004	02150	47301	41023	
	3					42	11170	41201	20436	22105	
83	41				37	77777	77777	77777	77777	77760	
101	50			1677	77777	77777	77777	77777	77777	77734	
	25			1770	73777	77777	71554	77777	77776	70774	
	20			1734	42260	36634	35312	16067	77653	36044	
	10			620	20714	00420	64141	30210	01470	20231	
	5			1000	60310	30220	00000	00220	60460	30004	
	4			1667	35170	05511	25702	53322	77032	10443	
107	53			37777	77777	77777	77777	77777	77777	77760	
131	65		377	77777	77777	77777	77777	77777	77777	77760	
	26			1634	45062	04144	04004	56032	04432	00042	36720
	13		1267	77723	45221	15105	11634	45045	44225	16277	77325
	10		6	52020	00200	22004	40100	66020	12043	06000	00030
	5		1727	52023	56373	62617	70360	77432	36763	56202	57274
139	69	4	77777	77777	77777	77777	77777	77777	77777	77777	77744
	46	4	76637	77731	75477	77777	77777	77775	47766	37777	54744
	23	4	72055	65424	32452	07006	07414	03412	45305	06566	41345
	6	1	57550	47372	56617	22064	32472	36641	62124	04335	10117
	3	3	50006	12434	15604	44500	03000	24444	16607	05214	00270

the companion register. The process can be carried out by systematic diagonalization.

As an example, we construct a primitive irreducible cubic using $z = x^9 + x^7 + x^6 + x^4 + 1$ (1321) for $p = 13$. We find $z^2 = x^{11} + x^{10} + x^9 + x^7 + x^6 + x^4 + x^3 + x^2$ and $z^3 = x^9 + x^7 + x^6 + x^5$. The calculations can be arranged:

$$\begin{array}{llll} 1\colon 000 & 000 & 000 & 001 \\ z\colon 001 & 011 & 010 & 001 \\ z^2\colon 111 & 011 & 011 & 100 \\ z^3\colon 001 & 011 & 010 & 000 \end{array} \qquad \begin{array}{llll} 1\colon 000 & 000 & 000 & 001 \\ z^2+z\colon 000 & 000 & 000 & 001 \end{array}$$

$$z^3 + z + 1\colon 000 \quad 000 \quad 000 \quad 000.$$

Specifically, z^2 is used (vacuously) to clear the first column; then discarded. The second column is now empty; z^3 is used to clear the third column and discarded. All but the last column are now clear and $z^3 + z$ is used to finish the job; the polynomial is $z^3 + z + 1$. Alternatively we could have started from 1 and the right hand column:

$$\begin{array}{llll} z+1\colon 001 & 011 & 010 & 000 \\ z^2\colon 111 & 011 & 011 & 100 \\ z^3\colon 001 & 011 & 010 & 000, \end{array}$$

$$\begin{array}{llll} z+1\colon 001 & 011 & 010 & 000 \\ z^3\colon 001 & 011 & 010 & 000 \end{array} \qquad \begin{array}{llll} z^3+z+1\colon 000 & 000 & 000 & 000. \end{array}$$

The diagonalization process, while requiring large storage, is very rapid. No more than $2pn^2$ additions need be performed.

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Improved Methods to Calculate the Characters of the Symmetric Group

By Stig Comét

1. Introduction. It was shown by Bivins and others [1] how the characters χ_κ^ρ of the symmetric group of degree N may be calculated with the aid of an electronic computer. By repeated use of a recursion formula due to Murnaghan (ref. in [1]), the characters are finally expressed by means of special characters, with $\kappa = 1^N$, for which an explicit formula exists, due to Frobenius. The same principle was applied in a program written for the electronic computer BESK at Stockholm. As was indicated at an early stage [2], I have used the Nakayama version of the recursion formula in this program, thus avoiding some of the trials giving zero terms. Aiming at economy with respect to storage space, in order to permit as high values of N as possible, I tried different ways of representing partitions binarily; some of them were described in [3]. The one which was then called Second Method turned out to be most convenient. Some details of the program are described in §4 as an introduction to the use of this method. The general plan of the calculation is given in §3. Some observations from the runnings are reported and briefly discussed in §5.

It should be mentioned here that the program has been run at BESK, free of cost, during periods when machine time has been available. I am very grateful to Matematikmaskinnämnden, The Swedish Board for Computing Machinery, for this favor.

The employed notation for partitions is a useful tool even for certain theoretical considerations. This fact was developed recently [4], and we shall quote, in §6, some of the results obtained. Further, in §7 and §8, we shall deduce a few formulas, expected to simplify the character calculations.

2. Definitions, Notations, and Formulas. We denote by χ_κ^ρ a simple character of the symmetric group of degree N, remembering that this character is completely identified by two partitions, ρ and κ, of the integer N. One of them, written as a subscript, indicates the class of conjugate group elements to which the character corresponds, the other one, written as a superscript, determines the irreducible representation from which the character emanates, as the trace of the representing matrix. In particular, the partitions $\rho = (N)$ and $\rho = (1^N)$ specify the unit and the alternating representations, respectively.

The class $\kappa = (1^N)$ consists of one element only, namely the unit element of the group. In a representation indicated by the partition $\rho = (r_1, r_2, \cdots, r_m)$, where $r_1 \geqq r_2 \geqq \cdots \geqq r_m \geqq 0$, the corresponding character is obtained from the Frobenius formula

$$\chi_{(1^N)}^{\rho} = \frac{N!}{R_1!R_2!\cdots R_m!} \prod_{(i<j)} (R_i - R_j) \tag{1}$$

where

$$R_i = r_i + m - i; \qquad i = 1, 2, \cdots, m. \tag{2}$$

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There is no closed formula giving the characters of an unspecialized class $\kappa = (k_1, k_2, \cdots, k_n)$, but, with the aid of Murnaghan's recursion formula

$$\chi_\kappa^\rho = \sum_{(i)} \epsilon_i \chi_\alpha^{\rho^{(i)}} \tag{3}$$

they may be expressed by means of characters of a symmetric group of lower degree. In (3), the partition κ' appears, if one of the components of κ, say k_a, is deleted. The partitions $\rho^{(i)}$ are obtained from ρ following certain rules, also telling us whether the sign factor ϵ_i has the value $+1$ or -1 or 0.

The method of hook removals, introduced by Nakayama [5], simplifies the application of (3) in two respects. Firstly, by this method only those $\rho^{(i)}$ are calculated which correspond to an $\epsilon_i \neq 0$, and, secondly, the value of ϵ_i is obtained as

$$\epsilon_i = (-1)^{z_i} \tag{4}$$

where z_i is the so-called leg-length of a hook.

It is advantageous to use, for the representation partitions ρ and $\rho^{(i)}$ in (3), the notation introduced as Second Method in [3]. From now on, we shall give to this notation the name of *rim* of the partition, because of its geometrical meaning, as in a paper by Frame and others [6]. The rim R of a partition $\rho = (r_1, r_2, \cdots, r_m)$ can be defined as a sequence of binary digits, in which each component r_i is represented by a digit = 0, immediately followed by $r_i - r_{i+1}$ digits = 1, considering r_{m+1} to be = 0. For example, the rim of $\rho = (6.6.5.5.3.2) = (6^2.5^2.3.2)$ is

$$R = 001001101011. \tag{5}$$

Conversely, the partition ρ is easily reconstructed from R, the component r_i being equal to the total number of 1-valued digits to the right of the ith zero. Evidently, 1-valued digits to the left, and zeros to the right, do not influence this determination of r_i, nor do they affect the definition of R. Therefore, a sequence of digits like

$$111110010011010110000000$$

is considered to be a rim of the same partition ρ as (5). We will distinguish (5) as the normalized form of the rim. In a computer as BESK, where the basic operations, assuming fixed binary point, act on numbers with absolute value ≤ 1, the rims are conveniently inserted immediately after the binary point. Thus, the rim is represented by a number

$$R = 1 - (1 + \sum_{i=1}^{m} 2^{R_i}) \cdot 2^{-R_0} \tag{6}$$

where R_i has the same meaning as in (2) and R_0 is an arbitrarily chosen integer (within the scope given by the word-length) satisfying $R_0 > R_1$. The normalized form of R corresponds to $R_0 = R_1 + 1$.

The geometrical illustration of R is obtained, if the digits are represented by adjacent line segments of equal lengths, those labeled 0 being directed vertically downwards, those labeled 1 horizontally to the left. The broken line constructed in this way is precisely the right, lower rim of the Young diagram of the partition ρ. Historically, the notation R was first based on this geometrical figure.

Each pair of digits, formed by a 0 in a rim and a 1 situated k steps in the rim

to the right of this 0, is said to form a *hook* of length k in the rim or, shorter, a k-hook. It corresponds to the Nakayama hook, and, together with the intermediate digits, it is a binary representation of the rim-hook used by Frame and others [6]. The two digits which form the first mentioned pair will be called the end digits of the hook. The number of zeros *between* the end digits is $=z_i$ in (4). The removal of a hook is effectuated by simply interchanging the two end digits of the hook.

In terms of the expression (6), these considerations will read as follows. The jth 0 of R, if counted from the binary point, may be used as left end digit of a k-hook, if and only if $R_j - k$ is different from all the quantities R_i . If this condition is fulfilled, the hook will be removed by substituting $R_j - k$ for R_j in that term of (6) where $i = j$. Thus, the value of R will change to R_+ where

$$R_+ = R + (1 - 2^{-k}) \cdot 2^{R_j - R_0}. \tag{7}$$

The number z_j of zeros between the end digits is equal to the number of indices i for which $R_j > R_i > R_j - k$.

Another operation used in the calculation process is the passage to the rim $\tilde{R}$ of the partition $\tilde{\rho}$, conjugate to ρ. This rim $\tilde{R}$ is obtained by writing the digits of R in the reversed order, replacing each 1 by 0 and each 0 by 1. Thus, corresponding to the example (5) we get

$$\tilde{R} = 001010011011$$

being the rim of $\tilde{\rho} = (6^2 \cdot 5 \cdot 4^2 \cdot 2)$.

It should be noted that the essential part of our notation is the sum

$$\sum_{i=1}^{m} 2^{R_i} = r \tag{8}$$

which enters in the expression (6). The special form (6) was chosen in order to adapt it to the operation list of BESK. It is clear that the form may be modified so as to suit other computers.

3. The Actual Calculation Program. In order to avoid complication, I have restricted the calculation to those representations ρ which can be stored, by means of their rims, within one word in BESK. This means the following restriction for technical reasons,

$$r_1 + m \leqq 38, \tag{9}$$

r_1 being the greatest component of ρ, as before. This condition indicates the limit of the capacity of the program in its present form.

If ρ varies through *all* partitions of N, we have $r_1 + m \leqq N + 1$. Hence, N must in such cases be limited to

$$N \leqq 37. \tag{10}$$

For an isolated ρ, satisfying (9), N may be greater. For instance, if $\rho = (19^{19})$, we get $N = 19^2 = 361$, the greatest value of N that can appear in the present version of the program.

The partition κ is stored in a different way (see below), not using the rim, but requiring that the components, k_a , of κ satisfy the condition

$$k_a \leqq 37 \quad \text{for all} \quad a = 1, 2, \cdots, n. \tag{11}$$

In fact, if a component of κ is $\geqq r_1 + m$, the value of χ_κ^ρ is 0, and there is no need for calculation. Thus, the condition (11) means no further restriction, from a practical point of view.

Originally, the program was not intended to calculate complete character tables, but only to give individual characters, thus serving as a substitute for a character table. A set of compiling routines, activated by different start operations, has later been added, permitting the program to work in the following variants:

(i) Calculate all χ_κ^ρ for a given N, under the heading of one ρ at a time, either for strictly all partitions ρ or for only one in each pair of ρ and $\tilde{\rho}$. Stop after the completion of an N, and then, after start, continue with the next, $N + 1$.

(ii) As in (i), but under the heading of one κ, instead of ρ, at a time. In the case when χ_κ^ρ is not calculated because $\chi_\kappa^{\tilde{\rho}}$ has already been, the letter C is printed (for "conjugate").

(iii) Calculate all χ_κ^ρ for one given ρ.

(iv) Calculate all χ_κ^ρ for one given κ.

(v) Calculate χ_κ^ρ for one given pair of ρ and κ.

(vi) As in (iv) but, instead of giving out the values of χ_κ^ρ, select the one having the greatest absolute value. Print this χ_κ^ρ together with the corresponding ρ and κ.

The central, common part of these programs will now be outlined, and a few points of it will be treated in greater detail. Suppose that ρ and κ are already represented in the computer, ρ by means of its rim R, inserted, in its normalized form, in a storage cell, while κ is represented in the following way. Let us write

$$\kappa = (k_1, k_2, \cdots, k_n, 1^q) \tag{12}$$

where

$$k_1 + k_2 + \cdots + k_n + q = N.$$

The order of succession of the components k_a $(a = 1, 2, \cdots, n)$ must be fixed before the calculation, not necessarily a monotonous order although it is appropriate to begin with the greatest components. There may be some $k_a = 1$, but in most cases all the 1-valued components are comprised in 1^q. Now, each of the components k_a will be stored as represented by a number 2^{-40+k_a}. For the storage of these numbers a set of cells are at disposal, which we will call H_a. The number q is separately stored.

The calculation method consists in a repeated use of the recurrence formula (3), removing successively hooks of length $k_1, k_2, \cdots, k_n$ by the rule (7). After this, it remains to make calculations using (1) with q for N. The rims of the representation partitions appearing after the hook removals will be stored in a set of cells, called W_a $(a = 1, 2, \cdots, n)$, where W_a will contain a rim from which a k_a-hook is the next to be removed. Its sign position will contain the sign factor resulting from the preceding hook-removals.

Designating by χ a quantity intended to get the end-value χ_κ^ρ, by P and P_+ two partition rims under consideration, and by e and e_+ certain corresponding sign factors, we describe the calculation:

1. Insert initial value 0 for χ.

If $n = 0$, *i.e.* if $\kappa = (1^N)$, put R for P_+ and pass directly to point 6 below.

If $n > 0$, insert R into W_1, put 1 for a, and arrange such an order modification that point 3 will not be neglected. Pass to point 2.

2. The contents of W_a are examined: The sign digit of this word replaces e, the rest of it replaces P. Pass to point 3 or 4 according to latest order modification.
3. The positions j of the left end-digits of all possible k_a-hooks in P are determined. Digits = 1, placed in the found positions, are called *hook indicators* and are inserted into H_a (together with the number 2^{-40+k_a} already stored there). Pass to point 4.
4. The contents of H_a are examined:
   If there are no hook indicators, change a to $a - 1$.
   If the new $a > 0$: After modification so as to neglect point 3, pass to point 2.
   If the new $a = 0$: Calculation ended. Pass to output routine.
   If there are one or more hook indicators: The most significant of them is removed from H_a. Corresponding to the same indicator, the k_a-hook is removed from P, giving P_+. The number, z, of zeros within this hook is counted, and $e \cdot (-1)^z$ gives e_+.
   If $a < n$: Pass to point 5.
   If $a = n$: Pass to point 6.
5. Insert e_+ into the sign position and P_+ into the other positions of W_{a+1}. Change a to $a + 1$. After order modification, assuring that point 3 will not be neglected, pass to point 2.
6. Calculate $\chi^{\tau}_{(1^p)}$ using formula (1) for the partition π corresponding to the rim $P_+ . -NB$. For all p such that $3 \leqq p \leqq M$ $(= 13)$, these characters are precalculated for all π and permanently stored.
   Replace χ by $\chi + e_+ \cdot \chi^{\tau}_{(1^p)}$. Pass to point 4.

4. Some Details of the Program. It should be mentioned that great values of p may cause $\chi^{\tau}_{(1^p)}$ and χ to exceed the word-length. Multiple-word routines are then automatically called into action, permitting, in the actual lay-out of the program, character values of at most 576 decimal digits.

Further, if $p > M' \geqq M$ $(M' = 19)$, the calculation of $\chi^{\tau}_{(1^p)}$ is not executed in point 6, but the rim P_+ is stored and then the program passes to point 4. For the storage of different P_+ and for accumulating coefficients of them, a set of working storage cells is available. When it occurs that this set is fully occupied, and always before passing to the output routine, the calculations of point 6 are executed for all the stored rims, and the value of χ is correspondingly determined. After clearing the set of working cells, the program may continue, if necessary.

Some of the points which have been programmed in a specific way will be described in the sequel.

a. Determination of the indicators (*point* 3). By shifting the rim P to the left k_a steps (and dropping the digits passing the binary point), a number Q is obtained. By changing all zeros of P into 1, and all ones into 0, one form of complement, $\bar{P}$, is obtained. The "logical product" of Q and $\bar{P}$ gets a 1 in those positions (only) where both Q and $\bar{P}$ have a 1. Thus, this logical product gives directly all the indicators of k_a-hooks in P, and it is inserted into H_a.

b. Test on hook indicators in H_a (*point* 4). The negative contents (modulo 2) of H_a are normalized. This means that a number, represented by binary digits, is shifted to the left, until unequal digits appear on either side of the binary point. In the actual case, the normalization will give a 1 to the left, a 0 to the right of the

binary point. If there were no indicators in H_s, there would be only zeros to the right of the binary point after the mentioned normalization. This case is very easily distinguished from the opposite one.

c. Removal of an indicator and a hook (*point* 4). The normalization process includes counting the shift steps. Thus, the number, v, of shift steps during the normalization under point b above determines the position of the left end-digit of the first k_s-hook, this position being the $(v + 1)$st after the binary point. The number $h = 2^{-v-1}$, obtained from 2^{-1} by shifting v steps to the right, is subtracted from the contents of H_s, thus removing the first indicator.

Further, if the negative number $-h$ is shifted k_s steps to the right and then h is added, we obtain

$$h' = h - h \cdot 2^{-k_s} = 2^{-v-1} - 2^{-k_s-v-1}.$$

According to (7), the rim, appearing when the hook in question is removed from P, is obtained as $P_+ = P + h'$.

d. The number of zeros within the hook (*point* 4). The logical product of the numbers h' and P consists of those digits of P which appertain to the hook, and, outside the hook, zeros. By means of a $v + 1$ steps shift to the left, it is brought to such a position that the interior of the hook begins at the binary point. In the present version of the program, the number, z, of zeros within the hook is now obtained by repeated normalizations, accumulating only the number of those shift steps when zeros pass the binary point. It would have been better to examine the hook one digit at a time, because such a process is more rapid for hooks of short length. Removals of short hooks have to be repeated considerably more times than those of longer hooks.

Point 4 is the most frequently repeated sequence of the program in the phase of applying the recursion formula (3). The programming methods just described have resulted from my efforts to reduce the execution time for this sequence as far as possible. The weakest point is still the determination of z.

With respect to the definition (12), removal of hooks of length = 1 need not occur in a standard running of the program, unless such hooks are used for special purposes, e.g. checking. Instead of the removal of 1-hooks, the Frobenius formula (1) is used which, fortunately, works considerably faster.

e. The calculation of $\chi^{\pi}_{(1^n)}$ (*point* 6). Only a few remarks on the programming of formula (1) are to be made. The calculated characters, which are all integers, may have very large values requiring arrangements for multiple-word calculation. Further, the numerator, being still larger, should not be explicitly evaluated, and, finally, operations which might introduce round-off errors should be avoided. The following strategy will meet these demands. Each factor entering in (1) is factorized into a product of prime powers. In a set of storage cells, one cell for each prime number, the exponents of the prime powers are accumulated, i.e. added if they come from the numerator, subtracted if they come from the denominator. Afterwards, the character is obtained by successive multiplications of the prime numbers, with due regard to the final values of the exponents, which will never be negative. When multiple word-length is required, the number of storage cells is successively adapted to the actual demand. As an intermediate stage, as many of the prime factors as possible without exceeding one word-length are multiplied, thus reducing the required number of multiple-word operations.

5. Observations from the Computing Work. The version (i) of the program (see §3) has been run by the regular staff of BESK-operators, when computer time has been available. Complete character tables have been calculated for all $N \leqq 20$. The results were produced by the machine punched on paper tapes, and these have also been printed out on paper sheets [8]. In each pair of conjugate representations, ρ and $\bar{\rho}$, only one has been treated. The chosen representation partition has been written out as a common heading for the characters in this representation. These have then been listed following the class partitions, κ, which succeed each other in lexicographical, non-increasing order, beginning with $\kappa = (N)$ and ending with $\kappa = (1^N)$. The calculation time, including the tape punching of the results, was for the table $N = 16$ about 105 minutes, for the table $N = 20$ about 51 hours, of course scattered on many partial runnings.

The versions (ii) through (v) have not been systematically employed for table calculations. Yet, many series of experiments have been made, giving some information on the differences in complexity between the calculation schemes in different cases, *e.g.* for different classes. The results can not be reported here.

The variant (vi) was prepared in view of giving an experimental answer to the question proposed in Note 9 of the paper [1] by Bivins and others. The case of $\kappa = (1^N)$ has been treated for all $N \leqq 30$, and we will list here those representation partitions which were found to give the greatest character for each of the $N = 3, 4, \cdots, 30$: (2.1), (3.1), (3.1^2), $(3.2.1)$, $(4.2.1)$, $(4.2.1^2)$, $(4.2^2.1)$, $(4.3.2.1)$, $(5.3.2.1)$, $(5.3.2.1^2)$, $(5.3.2^2.1)$, $(6.4.2.1^2)$, $(5.4.3.2.1)$, $(6.4.3.2.1)$, $(6.4.3.2.1^2)$, $(7.4.3.2.1^2)$, $(7.5.3.2.1^2)$, $(7.5.3.2^2.1)$, $(7.5.3.2^2.1^2)$, $(7.5.4.3.2.1)$, $(7.5.4.3.2.1^2)$, $(8.5.4.3.2.1^2)$, $(8.6.4.3.2.1^2)$, $(8.5.4.3.2^2.1^2)$, $(8.6.4.3.2^2.1^2)$, $(8.6.4.3^2.2.1^2)$, $(8.6.5.4.3.2.1)$, $(8.6.5.4.3.2.1^2)$.

Giant character values may appear already at rather small degrees N. As an example, for $N = 30$, the maximum character is a number with 16 decimal digits. The greatest value calculated by the version (v), under the condition (9), is expressed by 325 decimal digits; it corresponds to $N = 361$, $\rho = (19^{19})$, $\kappa = (1^{361})$. Another character, corresponding to the same N and ρ, but $\kappa = (7.5.1^{3.9})$, is a 313-digit, negative number, *etc.*

There is no particular checking sequence in the program. The reason is that the main purpose of the program is to calculate individual characters, as in version (v), and then the normal checking method, based on orthogonality, is of no use. Therefore, in order to check a calculated value, χ_κ^ρ, one should recalculate it in introducing the components $k_1, k_2, \cdots$ of κ in an *altered order*, this leading to a check if they are not all equal. If they are all $= 1$, *i.e.* $\kappa = (1^N)$, the recalculation is made with one or more $k_a = 1$ in (12), where q is adequately diminished. Thus, the only unchecked cases are those of $\kappa = (p^q)$ with $p > 1$. Concerning these cases, see, however, §7.

Most of the tables calculated by version (i) being unchecked, it is still possible to subject their values, as they are punched on paper tape, to the orthogonality test by means of a separate program. However, it will be neither possible nor necessary to carry out this test to its full extent.

6. Deductions for Further Improvements. Although it is interesting that, theoretically, the entire character calculation could be built up by hook removals

it is important that, in practice, more effective routines can be introduced, reducing the number of hook removals. For such a purpose, the Frobenius formula (1) was included in the program described above, where it could replace the removals of all hooks of length = 1. In an improved program, this formula is to be generalized to the case of $\kappa = (p^q)$ where p may be > 1.

Another way of reducing the number of hook removals to be executed is to observe that one and the same rim may be obtained several times when sequences of hooks of length $k_1, k_2, \cdots, k_{n'}$ $(2 \leqq n' < n)$ are removed. Instead of calculating the contribution to χ of that rim each time it occurs, one should multiply the contribution by an appropriate coefficient. Already in the described version of the program, this procedure was used at a certain point, as indicated in the beginning of §4 ($\cdots$ if $p > M' \geqq M \cdots$). It should, however, be used more systematically.

The realization of these desiderata requires some new formulas and rules. These can be based on the rim notation. In [4], detailed foundations and deductions are given, and, in the present paragraph, we shall quote from [4] what is needed for the developments in §7 and §8.

The essential part of the rim notation, *i.e.* the sum (8):

$$r = \sum_{i=1}^{m} 2^{R_i} \tag{8}$$

will now be called, as in [4], the *binary model* of the partition ρ. If (8) is written down in binary digits, to each 1-valued digit is associated a quantity called its *weight*, defined as the total number of zeros to the right of it up to the binary point. The weight of r is defined as the sum of the weights of its 1-valued digits. Calling it $N(r)$ we get, with regard to (2),

$$N(r) = r_1 + r_2 + \cdots + r_m . \tag{13}$$

Thus, the weight of r is equal to the number N of which ρ is a partition. A special, binary model called u, with $R_i = m - i$, has the weight

$$N(u) = 0; \qquad u = \sum_{i=1}^{m} 2^{m-i}. \tag{14}$$

The digits, 1 and 0, of a binary model r could be considered as indicating the presence or absence, respectively, of m movable objects at the locations represented by the digit positions of the number r.

The removal of a k-hook from the rim R, as in (7), is equivalent to an operation called *reduction* of the binary model r to another one, r', such that

$$r' = r + 2^{R_j - k} - 2^{R_j}. \tag{15}$$

It is easily proved (see [4] for this and other details) that

$$N(r') = N(r) - k,$$

i.e., k is the decrease in weight. We say that (15) is a reduction of r by the amount k. This reduction could be described by an operation on the jth object of r consisting in moving it k steps to the right.

A sequence of successive reductions by the amounts $k_1, k_2, \cdots, k_n$ is called a *chain* of reductions. If $k_1 + k_2 + \cdots + k_n = N$, the chain will transform r to a

binary model u of the form (14). The number of existing chains of reductions by the indicated amounts, leading from a given r to u, is not independent of the order of succession of the reduction amounts. Therefore, we have introduced the *value* of a chain as $(-1)^{z_1+z_2+\cdots+z_n}$, where z_j is the number of objects (1-valued digits) passed by the moved object during the reduction by the amount k_j. As shown in [4], the *sum of the values* of all chains of reductions by the amounts $k_1, k_2, \cdots, k_n$ is the same as the sum obtained, if another order of succession is chosen for these amounts. It follows that this sum is characterized by the *partition* κ of N having the components $k_1, k_2, \cdots, k_n$. It will be denoted by $\chi_\kappa(r)$ and sometimes called the characteristic value of κ determined by r. It is, in fact, identical with the character χ_κ^ρ, but it may be convenient to keep the notation $\chi_\kappa(r)$.

A generalization is obtained in considering those chains which reduce r, not to u, but to r', where r' is the binary model of a partition ρ' of a number $N(r') < N(r)$. The sum of the values of all chains of reductions leading from r to r' and consisting of reductions by the amounts indicated by the components of a certain partition κ'' of the number $N(r) - N(r')$, will be denoted by $\chi_{\kappa''}(r; r')$ and called the characteristic value of κ'' determined by the reduction of r to r'. If κ' is a partition of the number N', we get the fundamental formula

$$\chi_{\kappa''\kappa'}(r) = \sum_{r'} \chi_{\kappa''}(r; r') \cdot \chi_{\kappa'}(r') \tag{16}$$

where the binary models r' correspond to partitions ρ' of the number N'. The demonstration depends mainly on the consideration of the chains leading from r to r' and of those leading from r' to u. Concerning the details, the reader is referred to [4]. It should be observed that (3) is contained in (16) as a special case.

We shall show, in §7, how (16) may simplify the calculation of χ_κ^ρ if special properties of the components of κ can be utilized, and, in §8, how (16) may lead to relations independent of κ, but depending on different ρ.

7. Utilization of Properties of κ. If κ is a partition of N, let κ'' be the partition formed by those components of κ which have a common factor p, where p is an integer > 1. Then, the components of κ'' may be written $pq_1, pq_2, \cdots, pq_h$. This κ'' is a partition of a number, say pM, also containing the factor p. The numbers $q_1, q_2, \cdots, q_h$ are the components of a partition of M, which we will call ξ. Finally, we put

$$N - pM = N'$$

being, in (16), the weight of r'. It is clear that κ' is a partition of this N', as it is formed by all those components of κ which are not comprised in κ''.

At the determination of $\chi_{\kappa''}(r; r')$, the objects of r are moved to the right by multiples of p steps. Therefore, the binary model r may be split into p binary models $r^{(1)}, r^{(2)}, \cdots, r^{(p)}$ in such a way that each object remains within one and the same model when moved. In order to get suitable and unambiguous rules, we shall assume that both the number of 1-valued digits of r and the total number of digits of r are divisible by p. In fact, we can always adjoin ones to the right of r and zeros to the left, because this does not change the weight of r. Now, the splitting of r is most easily accomplished in writing the digits of r in successive, vertical columns of

height p, as in a matrix. The digits found in horizontal rows then form the binary models $r^{(j)}$, $j = 1, 2, \cdots, p$.

Each chain of reductions corresponding to κ'', executed on r, is represented by chains of reductions by the amounts $q_1, q_2, \cdots, q_h$ executed on the binary models $r^{(j)}$. Thereby, the reduction amounts on $r^{(j)}$ will be denoted $q_1^{(j)}, q_2^{(j)}, \cdots, q_{h_j}^{(j)}$, which are the components of a partition $\xi^{(j)}$ of a number $M^{(j)}$. It is necessary that

$$M^{(j)} \leqq N(r^{(j)}). \tag{17}$$

Because $M^{(1)} + M^{(2)} + \cdots + M^{(p)} = M$, it follows that necessarily

$$\sum_{j=1}^{p} N(r^{(j)}) \geqq M, \tag{18}$$

as soon as there are chains corresponding to κ''.

On the contrary, if

$$\sum_{j=1}^{p} N(r^{(j)}) < M, \tag{19}$$

there can not be any chains corresponding to κ'', and, hence, the sum in (16) is empty. Consequently, if (19) holds, the character $\chi_{\kappa''\kappa'}(r) = 0$.

Returning to the case when (18) is satisfied, we have to distribute the components of ξ, in all possible ways, to partitions $\xi^{(j)}$ of numbers $M^{(j)}$ satisfying (17). For each of these distributions we subject the binary models $r^{(j)}$ to chains of reductions by the amounts given by the appropriate $\xi^{(j)}$. We denote the resulting binary models by $r'^{(j)}$. One particular system of such binary models is obtained by means of chains of reductions which correspond to certain chains of reductions executed on r. The binary model r', obtained by this reduction of r, can be written down very easily, if the digits are read column by column in the matrix where the $r'^{(j)}$ are rows.

Further we may conclude, similarly to the discussion in [4] (p. 105), that the value of the chain from r to r' differs from the product of the values of the chains from $r^{(j)}$ to $r'^{(j)}$ only by a factor $= \operatorname{sgn} Q \cdot \operatorname{sgn} Q'$, where $\operatorname{sgn} Q$ is $+1$ or -1 depending on r and the $r^{(j)}$ only, whereas $\operatorname{sgn} Q'$ is $+1$ or -1 depending on r' and the $r'^{(j)}$ only. For the sums of these values we get

$$\chi_{\kappa''}(r; r') = \operatorname{sgn} Q \cdot \operatorname{sgn} Q' \sum \prod_{j=1}^{p} \chi_{\xi^{(j)}}(r^{(j)}; r'^{(j)}) \tag{20}$$

where the summation is extended over all distributions of the components of ξ to such $\xi^{(j)}$ which lead to the actual set of $r'^{(j)}$. If some of the components of ξ are equal, it may happen that one set of $\xi^{(j)}$ is obtained several times. Let a component, valued i, be present τ_i times in ξ and τ_{ji} times in $\xi^{(j)}$. This set of $\xi^{(j)}$ is obtained so many times as given by the expression

$$\varphi(\xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(p)}) = \prod_{i=1}^{M} \frac{\tau_i !}{\prod_{j=1}^{p} \tau_{ji} !} \tag{21}$$

We can now write (20) in the following form:

$$\chi_{\kappa''}(r; r') = \operatorname{sgn} Q \cdot \operatorname{sgn} Q' \cdot \sum \varphi(\xi^{(1)}, \cdots, \xi^{(p)}) \prod_{j=1}^{p} \chi_{\xi^{(j)}}(r^{(j)}; r'^{(j)}) \tag{22}$$

where the summation is extended over all *different* distributions of ξ to $\xi^{(j)}$. When inserting (22), with (21), into (16), we observe that the factor sgn Q, but not the factor sgn Q', may be put outside the summation sign (over r').

The sign factor sgn Q is determined by means of the number ω of inversions of the objects of r introduced by replacing r by the set $r^{(j)}$. A method to obtain ω, suitable for a computer program, could be sketched as follows: The method consists in counting over the objects column by column in the matrix formed by $r^{(j)}$. Let, at a certain moment, $m^{(j)}$ be the number of still uncounted objects in the jth row. If, then, the next object is encountered in the Jth row, the following two operations are made:

(i) The accumulated value of ω is augmented by $m^{(1)} + m^{(2)} + \cdots + m^{(J-1)}$.

(ii) $m^{(J)}$ is replaced by $m^{(J)} - 1$.

When all objects have been counted, we get

$$\operatorname{sgn} Q = (-1)^{\omega}. \tag{23}$$

Of course, sgn Q' is obtained similarly from $r'^{(j)}$.

Special cases occur if (18) reads as an equality. In these cases, the relations (17) must be equalities, too, and it follows that all the binary models $r'^{(j)}$ have the weight $= 0$. Therefore, there is only one set $r'^{(j)}$ and one single r'. This r' is the binary model of the partition called by Littlewood [7] the p-residue, and by Nakayama [5] the p-core of ρ. Observing that, in these cases, the characteristic values $\chi_{\xi^{(j)}}(r^{(j)}; r'^{(j)})$ become veritable characters, the formula (16), with regard to (22) and (21), gives the following expression:

$$\chi_{\kappa''\kappa'}(r) = \operatorname{sgn} Q \cdot \operatorname{sgn} Q' \cdot \chi_{\kappa'}(r') \cdot \sum \varphi \prod_{j=1}^{p} \chi_{\xi^{(j)}}(r^{(j)}). \tag{24}$$

This will be still more simplified if *all* components of κ have the common factor p. In this case κ'' may be chosen $= \kappa$ so that κ' will vanish. If all the binary models $r^{(j)}$ contain equal numbers of 1-valued digits, the character $\chi_\kappa(r)$ is given by (24), where $\chi_{\kappa'}(r')$ has to be put $= 1$. If the $r^{(j)}$ do not satisfy this condition (although the number of 1-valued digits in r is divisible by p), the inequality (19) holds (see [4]) and, consequently, the character $\chi_\kappa(r)$ is $= 0$.

Finally, if $\kappa = (p^M)$, the last considerations may be completed by giving the explicit expressions for $\chi_{\xi^{(j)}}(r^{(j)})$. Because $\xi^{(j)} = (1^{M^{(j)}})$, the Frobenius formula (1) is applicable. Writing, in analogy to (2),

$$R_h^{(j)} = r_h^{(j)} + m - h \qquad (h = 1, 2, \cdots, m)'$$

where m is the number of 1-valued digits in each $r^{(j)}$, we get for the quantities in (21):

$$\tau_1 = M; \qquad \tau_{j1} = M^{(j)}; \qquad \tau_i = \tau_{ji} = 0 \qquad \text{for } i \neq 1.$$

Observing that there is only one set of $\xi^{(j)}$, we obtain from (24) the non-zero characters:

$$\chi_{(p^M)}(r) = \operatorname{sgn} Q \cdot \operatorname{sgn} Q' \cdot M! \prod_{j=1}^{p} \left(\frac{\prod_{h<i} (R_h^{(j)} - R_i^{(j)})}{R_1^{(j)}! \cdots R_m^{(j)}!} \right) \tag{25}$$

In summarizing, we find that the utility of the described method may appear in two respects. Firstly, the binary model r is split into the models $r^{(j)}$ which, corresponding to partitions of smaller numbers than the original one, will need considerably less computing work. Secondly, the calculation of the factor φ by means of (21) will shorten the computing work as soon as it replaces the treatment of several equal terms in a sum. The utility of the inequality (19) for the detection of zero-valued characters is evident.

8. Relations Independent of κ. Turning our attention to the binary models r and r' in $\chi_{\kappa''}(r; r')$ of (16), we write, as in (8),

$$r = \sum_{i=1}^{m} 2^{R_i} \quad \text{and} \quad r' = \sum_{i=1}^{m} 2^{R_i'} \tag{26}$$

where, as in (2),

$$R_i = r_i + m - i \quad \text{and} \quad R_i' = r_i' + m - i, \tag{27}$$

the number of 1-valued digits being the same, m, in both models.

Preliminarily, let us assume that r and r' satisfy the conditions:

$$\begin{cases} r_i \geqq r_i' & \text{for } i = 1, 2, \cdots, m, \\ r_i' \geqq r_{i+1} & \text{for } i = 1, 2, \cdots, m-1. \end{cases} \tag{28}$$

By (27), this means

$$R_i \geqq R_i' \quad \text{and} \quad R_i' > R_{i+1}, \text{ respectively.}$$

It follows that, by each chain of reductions leading from r to r', the ith object of r will become the ith object of r', *i.e.* the objects will not be permuted. The values of such chains, corresponding to the components of κ (written instead of κ''), will be $+1$, and the sum, $\chi_\kappa(r; r')$, of these values will be equal to the number of the chains.

If the components of κ are $k_1, k_2, \cdots, k_n$, with

$$k_1 + k_2 + \cdots + k_n = N,$$

each chain is constructed by distributing these components to sequences $k_1^{(i)}, k_2^{(i)}, \cdots, k_{n_i}^{(i)}$ $(i = 1, 2, \cdots, m)$, the ith sequence indicating the movements of the ith object, thus

$$k_1^{(i)} + k_2^{(i)} + \cdots + k_{n_i}^{(i)} = R_i - R_i' = r_i - r_i'. \tag{29}$$

The chains depend merely on κ and on the differences $r_i - r_i' = d_i$, but not on the components r_i and r_i' themselves, provided that they satisfy (28). The number of chains is equal to the number of distributions of κ satisfying (29). If another pair of binary models r and r' satisfies (28) and gives the same set of differences d_i, which may, however, come in a different order, the system (29) will still have the same number of solutions. Therefore, denoting by δ the partition of N with the components $d_1, d_2, \cdots, d_m$, we find that the number of chains leading from r to r' depends merely on κ and δ. We will write this number ψ_κ^δ, and for binary models r and r' satisfying (28) we have

$$\chi_\kappa(r; r') = \psi_\kappa^\delta$$

In the general case, the condition (28) does not hold but only

$$r_i \geqq r_i' \quad \text{for} \quad i = 1, 2, \cdots, m, \tag{30}$$

expressing that the objects never move to the left during a reduction. The objects may be permuted by a reduction chain. Let us consider those chains by which the ith object of r becomes the p_ith object of r' $(i = 1, 2, \cdots, m)$. Here, the numbers $p_1, p_2, \cdots, p_m$ form a permutation P of the integers $1, 2, \cdots, m$. Not all permutations P are possible, because it is required that

$$R_i - R'_{p_i} \geqq 0 \quad \text{for all} \quad i = 1, 2, \cdots, m. \tag{31}$$

We write

$$R_i - R'_{p_i} = d_i,$$

which are, as before, the components of a partition δ, now depending on P. The number of the chains considered is $\leqq \psi_\kappa^\delta$. It will be $< \psi_\kappa^\delta$ as soon as a distribution like (29) would cause two objects to coincide after some reduction steps, which is not permitted. By a reasoning, made in detail in [4] (see p. 99 ff.), we find that, if sgn P is defined as $+1$ for all even and -1 for all odd permutations, the sum of all sgn $P \cdot \psi_\kappa^\delta$ will be equal to the sum of all values of the chains leading from r to r'. Thus,

$$\chi_\kappa(r; r') = \sum_{(P)} \operatorname{sgn} P \cdot \psi_\kappa^\delta \tag{32}$$

where P ranges over all permutations of $1, 2, \cdots, m$ compatible with (31). This formula holds for all partitions κ of the number N, so that we may leave out the index κ. By $\chi(r; r')$ and ψ^δ we could mean vectors or 1-column matrices, simplifying (32) to

$$\chi(r; r') = \sum_{(P)} \operatorname{sgn} P \cdot \psi^\delta \tag{33}$$

We note that, for $r' = u$, the quantity $\chi_\kappa(r; r')$ means the character χ_κ^ρ. Thus, the vector χ^ρ is, by means of (33), expressed as a linear combination of the vectors ψ^δ, both ρ and δ being partitions of N. Now, as the different vectors χ^ρ are linearly independent, and as the number of the vectors ψ^δ is equal to that of χ^ρ, we conclude that the vectors ψ^δ also form a linearly independent system.

At the practical application of these formulas we have to calculate the quantities ψ_κ^δ. This could be done either by (29) or by counting chains of reductions from an r to an r', arbitrarily chosen but in accordance with (28). It is essential that no determinations of the sign $+$ or $-$ are required at this stage of the process. This means that the most time-consuming point in the calculation program (see §4, d) is avoided. At the insertion in (33), the factor sgn P is determined independently of the κ.

In certain situations one may prefer expressing the ψ^δ in terms of χ^ρ. This is done by solving the system (33), with $r' = u$, with respect to ψ^δ. The obtained expressions also present a theoretical interest. In fact, ψ_κ^δ is a compound character for the class κ of the symmetric group in a representation obtained in the following way. Let S be the group of all permutations of a set of N objects. Dividing this set of objects in subsets containing $d_1, d_2, \cdots, d_m$ objects, according to the partition

δ, we consider the subgroup S^δ formed by those permutations which only permute the objects within the subsets. The group S is divided in complexes, S_1, S_2, $\cdots$, S_h, induced by the subgroup S^δ. For each element $s \in S$ we have

$$s \cdot S_i = S_{i'}$$

Thus, S_1, $\cdots$, S_h form a basis for a representation of S by matrices. The trace of the matrix representing s can be shown to be $= \psi_\kappa^\delta$ where κ is the class to which s belongs. The demonstration must be omitted here. The above mentioned expression of ψ^δ in terms of χ^δ indicates directly the structure of the described representation.

Returning our attention to the computing work required by (16), we see how the formulas (32) and (33) permit the calculation to be performed in "blocks". This will be the more necessary, the more the degree of the symmetric group increases. An improved computer program for character calculation should be planned so as to combine in the most efficient way the methods described in §7 and §8. This program has not yet been constructed, and, in fact, it is necessary first to establish the purpose for which the program is to be used, because this has a great influence on the decision concerning efficiency.

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Tables of Values of the Modified Mathieu Functions

By E. T. Kirkpatrick

1. Introduction. Mathieu functions are encountered in physical problems involving elliptical boundaries. When the method of separation of variables is applied to the wave equation or the diffusion equation expressed in elliptical coordinates, there result [1, p. 170] the Mathieu and modified Mathieu equations,

$$\frac{d^2y}{du^2} + (a - 2q\cos 2u)y = 0 \tag{1}$$

$$\frac{d^2y}{du^2} - (a - 2q\cosh 2u)y = 0. \tag{2}$$

The notation used is that of Goldstein, Ince and McLachlan. Solutions to the modified Mathieu equation (2) can be assumed to be of the form

$$y = Ce_{2n}(u, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cosh 2ru \tag{3a}$$

$$y = Ce_{2n+1}(u, q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cosh (2r+1)u \tag{3b}$$

$$y = Se_{2n+1}(u, q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sinh (2r+1)u \tag{3c}$$

$$y = Se_{2n+2}(u, q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sinh (2r+2)u. \tag{3d}$$

When the above infinite hyperbolic series are substituted into equation (2), recurrence relationships may be derived for allowable values of the characteristic numbers a_{2n+p} ($p = 0$ or 1) for the even functions, or b_{2n+s} ($s = 1$ or 2) for the odd functions for a given value of q. Recurrence equations also give the allowable values of the Fourier coefficients $A_{2r+p}^{(2n+p)}$ and $B_{2r+s}^{(2n+s)}$.

The formulas for both the characteristic numbers and the Fourier coefficients associated with each type of solution [1, p. 29, 37] are given below:

For $y = Ce_{2n}(u, q)$

$$a_{2n} = \frac{2q^2}{a - 2^2 -} \frac{q^2}{a - 4^2 -} \frac{q^2}{a - 6^2 -} \cdots \frac{q^2}{a - (2r)^2 -} \cdots \tag{4a}$$

$$v_{2r-2} = \frac{q}{a - (2r)^2 - qv_{2r}}, \qquad r \geqq 2 \tag{4b}$$

$$v_0 = \frac{2q}{a - (2r)^2 - qv_2}, \qquad r = 1 \tag{4c}$$

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where

$$v_{2r-2} = \frac{A_{2r}^{(2n)}}{A_{2r-2}^{(2n)}}. \tag{4d}$$

For $y = Ce_{2n+1}(u, q)$

$$a_{2n+1} = 1 + q + \frac{q^2}{a - 3^2 -} \frac{q^2}{a - 5^2 -} \cdots \frac{q^2}{a - (2r+1)^2 -} \cdots \tag{5a}$$

$$v_{2r-1} = \frac{q}{a - (2r+1)^2 - qv_{2r+1}}, \qquad r \geqq 2 \tag{5b}$$

$$v_1 = \frac{a - 1 - q}{q}, \qquad r = 1. \tag{5c}$$

For $y_{2n+1} = Se_{2n+1}(u, q)$

$$b_{2n+1} = 1 - q + \frac{q^2}{a - 3^2 -} \frac{q^2}{a - 5^2 -} \cdots \frac{q^2}{a - (2r+1)^2 -} \cdots \tag{6a}$$

$$v_{2r+1} = \frac{q}{a - (2r+1)^2 - qv_{2r+1}}, \qquad r \geqq 2 \tag{6b}$$

$$v_1 = \frac{a - 1 + q}{q}, \qquad r = 1. \tag{6c}$$

For $y_{2n+2} = Se_{2n+2}(u, q)$

$$b_{2n+2} = 4 + \frac{q^2}{a - 4^2 -} \frac{q^2}{a - 6^2 -} \cdots \frac{q^2}{a - (2r)^2 -} \cdots \tag{7a}$$

$$v_{2r-2} = \frac{q}{a - (2r)^2 - qv_{2r}}, \qquad r \geqq 3 \tag{7b}$$

$$v_2 = \frac{a - 4}{q}, \qquad r = 2. \tag{7c}$$

As a consequence [1, p. 24] of the Goldstein-Ince normalization

(8) $\int_0^{2\pi} y^2\, du = \pi$, then for $y = Ce_{2n}(u, q)$, $\quad 1 = 2\,[A_0^{(2n)}]^2 + \sum_{r=1}^{\infty} [A_{2r}^{(2n)}]^2$

and for the remaining three types,

$$1 = \sum_{r=0}^{\infty} [A_{2r+1}^{(2n+1)}]^2 = \sum_{r=0}^{\infty} [B_{2r+1}^{(2n+1)}]^2 = \sum_{r=0}^{\infty} [B_{2r+2}^{(2n+2)}]^2. \tag{9}$$

By dividing equations (8) and (9) through by $[A_0^{(2n)}]^2$ there are obtained equations of the form

$$\left[\frac{1}{A_0^{(2n)}}\right]^2 = 2 + \left[\frac{A_2^{(2n)}}{A_0^{(2n)}}\right]^2 + \left[\frac{A_4^{(2n)}}{A_0^{(2n)}}\right]^2 + \left[\frac{A_6^{(2n)}}{A_0^{(2n)}}\right]^2 + \cdots \tag{10}$$

$$= 2 + v_0^2 + (v_2 \cdot v_0)^2 + (v_4 \cdot v_2 \cdot v_0)^2 + \cdots.$$

2. Computation of the Modified Mathieu Functions. For a given value of q, the characteristic number is found by using a trial, error and interpolation method

using the appropriate transcendental continued fraction of the form of equation (4a). The computation was generally started with the 21st term equal to zero, and then the 20th, 19th, etc., terms were computed in turn.

The v_{2r} are all known from equations of the type (4b) and (4c) above and therefore A_0 can be evaluated from equation (10). Equation (4d) will then give the value of the remaining A_{2r}. The value of the modified Mathieu function can then be computed by summing the appropriate hyperbolic series.

In [2], E. L. Ince has given tables for the characteristic numbers, Fourier coefficients and values of the Mathieu functions satisfying equation (1). The method used by Ince was to apply the above formulas; the same method was used by the author to evaluate the modified Mathieu functions satisfying equation (2). The numerical work was accomplished by using an IBM 650 Digital Computer and the Wolontis Interpretive System of coding. The characteristic numbers and the Fourier coefficients are the same for both the ordinary and the modified functions. However, the problem of summing the hyperbolic series is much more difficult than that of summing the trigonometric series. Whereas the cos $2ru$ term is bounded by plus or minus one, the sinh $2ru$ and cosh $2ru$ approach infinity as r tends to infinity. Therefore many more terms of the hyperbolic series must be considered before it is possible to generate a product $A_{2r}^{(2n)} \cosh 2ru$ which will be negligibly small.

As more terms are taken, difficulties in computation are encountered. The v_{2r} can be made as small as desired by choosing r sufficiently large. In most cases after the characteristic number had been found by using the appropriate transcendental continued fraction, the computation of the Fourier coefficients was started by letting $v_{40} = 0$, and then compute the v_{38}, v_{36}, $\cdots$ v_0, in turn. As these values were computed, they were stored and used to form the products v_0^2, $(v_2 \cdot v_0)^2$, etc., for equation (10). It is to be noted that the smallest number which can be computed using the Wolontis Interpretive System is 10^{-50}. At first consideration this may not appear to be a serious limitation. However, in order to obtain, for example, fifteen coefficients up to $A_{28}^{(2n)}$, from the relations $A_{2r+2} = v_{2r}A_{2r}$, the v_{26} must be available. For it to be available the product of $(v_{26} \cdot v_{24} \cdot \cdots \cdot v_0)^2$ must be greater than 10^{-50}, or the unsquared value must be greater than 10^{-25}. Except for v_0, v_2 and v_4 which may sometimes be greater than one, all the remaining values are less than one. In fact, for convergence, $v_{2r} \to 0$ as $r \to \infty$. If, for example, the "average" size of the v_{2r} is of the order of 10^{-2}, only twelve v_{2r} can be multiplied together before their product becomes less than 10^{-25}. Therefore, for this example, the maximum number of coefficients that can be generated will be twelve or thirteen. A machine error stop was avoided by counting the products as they were formed and at the same time checking their magnitude. The maximum number of coefficients generated was twenty; that is, up to and including coefficient $A_{38}^{(2n)}$. The number of coefficients generated will depend on the values of q and order $2n$ being considered.

3. Description of Computer Programs. A computer program was developed for the IBM 650 Digital Computer which will compute the characteristic numbers and Fourier coefficients and will sum the appropriate series in order to find the value of either the ordinary or the modified Mathieu function. A complete description of this program titled "A Program for the Computation of Mathieu and Modified Mathieu Functions" is available from the author and will be published separately.

In order to ensure five figure accuracy for parameters in the ranges $0 \leqq q \leqq 20$, $0 \leqq n \leqq 3$ and $0 \leqq u \leqq 1.0$, the program can use a maximum of twenty terms. For parameters outside the given ranges, the accuracy is unknown. However, the user can estimate the accuracy by noting the magnitude of the last term added to the series.

4. Discussion of Tables. Tables 1–12 are provided herein for values of the four types of modified Mathieu function $Ce_{2n}(u, q)$, $Ce_{2n+1}(u, q)$, $Se_{2n+1}(u, q)$ and $Se_{2n+2}(u, q)$, for the ranges $q = 1(1)10(2)20$, $u = 0.1(0.1)1.0$ and $n = 0, 1, 2$. The Fourier coefficients computed in order to obtain these values were compared with existing tables [1] for over 150 values, and no discrepancy was found in the first seven digits of any coefficient generated. Also, the magnitude of the last term added in the series was noted to ensure the accuracy of the last position published.

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TABLE 1

Values of Modified Mathieu Function $Ce_0(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	.38010	.36588	.34209	.30863	.26554
2	.19704	.18047	.15340	.11679	.07224
3	.10989	.09505	.07142	.04076	.00576
4	.06564	.05329	.03412	.01034	−.01491
5	.04130	.03127	.01612	−.00179	−.01932
6	.02702	.01895	.00709	−.00623	−.01807
7	.01822	.01174	.00249	−.00734	−.01515
8	.01259	.00738	.00017	−.00705	−.01206
9	.00887	.00468	−.00094	−.00624	−.00931
10	.00636	.00298	−.00142	−.00529	−.00705
12	.00341	.00118	−.00153	−.00355	−.00388
14	.00190	.00042	−.00126	−.00229	−.00203
16	.00110	.00010	−.00095	−.00144	−.00101
18	.00065	−.00002	−.00069	−.00089	−.00045
20	.00040	−.00007	−.00049	−.00054	−.00016
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	.21308	.15200	.08378	.01097	−.06241
2	.02222	−.02961	−.07835	−.11775	−.14074
3	−.02984	−.06129	−.08309	−.08980	−.07753

TABLE 1 (*Continued*)

4	−.03764	−.05332	−.05766	−.04792	−.02464
5	−.03275	−.03843	−.03382	−.01887	.00267
6	−.02529	−.02528	−.01712	−.00270	.01269
7	−.01841	−.01551	−.00677	.00478	.01379
8	−.01290	−.00880	−.00094	.00728	.01122
9	−.00876	−.00443	.00199	.00722	.00773
10	−.00576	−.00172	.00319	.00606	.00457
12	−.00219	.00073	.00314	.00319	.00054
14	−.00055	.00127	.00211	.00113	−.00092
16	.00011	.00112	.00116	.00007	−.00106
18	.00032	.00081	.00051	−.00033	−.00074
20	.00034	.00052	.00014	−.00039	−.00038

TABLE 2

Values of Modified Mathieu Function $Ce_1(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	.85596	.85372	.84884	.83954	.82334
2	.67799	.66081	.63071	.58567	.52324
3	.50011	.47318	.42746	.36215	.27703
4	.35310	.32216	.27089	.20034	.11323
5	.24612	.21525	.16523	.09889	.02144
6	.17252	.14384	.09841	.04040	−.02332
7	.12249	.09685	.05717	.00848	−.04153
8	.08824	.06581	.03192	−.00795	−.04594
9	.06445	.04509	.01652	−.01561	−.04369
10	.04769	.03109	.00721	−.01840	−.03863
12	.02697	.01493	−.00151	−.01739	−.02697
14	.01582	.00713	−.00408	−.01366	−.01732
16	.00955	.00329	−.00433	−.00994	−.01054
18	.00591	.00138	−.00379	−.00696	−.00612
20	.00373	.00044	−.00306	−.00475	−.00337
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	.79697	.75652	.69758	.61571	.50717
2	.44103	.33753	.21334	.07268	−.07467
3	.17349	.05584	−.06712	−.18088	−.26516
4	.01512	−.08442	−.17102	−.22623	−.23096
5	−.05858	−.12880	−.17365	−.17745	−.13069
6	−.08277	−.12515	−.13706	−.10917	−.04351
7	−.08263	−.10328	−.09387	−.05217	.01053
8	−.07238	−.07784	−.05696	−.01349	.03573
9	−.05914	−.05494	−.02954	.00899	.04194
10	−.04621	−.03648	−.01097	.01974	.03785
12	−.02564	−.01254	.00704	.02185	.02025
14	−.01264	−.00112	.01088	.01463	.00571
16	−.00526	.00325	.00923	.00728	−.00172
18	−.00142	.00421	.00625	.00231	−.00401
20	.00037	.00374	.00359	−.00032	−.00369

TABLE 3

Values of Modified Mathieu Function $Ce_2(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	1.0988	1.1373	1.2008	1.2883	1.3974
2	1.0549	1.0724	1.0988	1.1297	1.1585

TABLE 3 (*Continued*)

3	0.96379	0.96299	0.95773	0.94205	0.90754
4	0.85071	0.83405	0.80203	0.74858	0.66614
5	0.72581	0.69617	0.64300	0.56150	0.44693
6	0.59925	0.56021	0.49256	0.39375	0.26316
7	0.48147	0.43689	0.36159	0.25581	0.12340
8	0.37974	0.33303	0.25593	0.15146	0.02752
9	0.29657	0.25032	0.17560	0.07791	−0.03165
10	0.23086	0.18674	0.11699	0.02902	−0.06385
12	0.14063	0.10314	0.04636	−0.01995	−0.08086
14	0.08716	0.05686	0.01298	−0.03411	−0.07028
16	0.05511	0.03123	−0.00175	−0.03393	−0.05319
18	0.03549	0.01694	−0.00746	−0.02879	−0.03736
20	0.02324	0.00892	−0.00894	−0.02268	−0.02491
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	1.5243	1.6624	1.8011	1.9253	2.0134
2	1.1753	1.1676	1.1197	1.0143	0.83594
3	0.84378	0.73947	0.58490	0.37624	0.12180
4	0.54718	0.38688	0.18749	−0.03624	−0.25175
5	0.29722	0.11677	−0.07877	−0.25853	−0.37664
6	0.10531	−0.06602	−0.22436	−0.33044	−0.34055
7	−0.02480	−0.16773	−0.27310	−0.30268	−0.22806
8	−0.10013	−0.20621	−0.25818	−0.22648	−0.10465
9	−0.13427	−0.20357	−0.21119	−0.14039	−0.00653
10	−0.14150	−0.17901	−0.15482	−0.06588	0.05530
12	−0.11743	−0.11198	−0.05854	0.02440	0.08984
14	−0.08057	−0.05594	−0.00279	0.05087	0.06479
16	−0.04921	−0.02021	0.02043	0.04520	0.02999
18	−0.02702	−0.00098	0.02511	0.02973	0.00504
20	−0.01284	0.00753	0.02147	0.01512	−0.00745

TABLE 4

Values of Modified Mathieu Function $Ce_3(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	1.1052	1.2213	1.4222	1.7189	2.1258
2	1.1586	1.2504	1.4052	1.6250	1.9097
3	1.1812	1.2486	1.3585	1.5061	1.6816
4	1.1654	1.2097	1.2780	1.3609	1.4428
5	1.1197	1.1428	1.1740	1.2010	1.2055
6	1.0538	1.0580	1.0563	1.0356	0.97750
7	0.97392	0.96113	0.93084	0.87011	0.76270
8	0.88402	0.85635	0.80166	0.70851	0.56459
9	0.78762	0.74742	0.67297	0.55533	0.38747
10	0.68880	0.63876	0.54956	0.41562	0.23594
12	0.50147	0.44023	0.33678	0.19314	0.02033
14	0.34898	0.28634	0.18502	0.05393	−0.08713
16	0.23815	0.18014	0.09012	−0.01830	−0.12097
18	0.16205	0.11131	0.03578	−0.04848	−0.11669
20	0.11083	0.06792	0.00673	−0.05603	−0.09738
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	2.6602	3.3395	4.1775	5.1777	6.3233
2	2.2548	2.6466	3.0569	3.4362	3.7080
3	1.8671	2.0337	2.1383	2.1234	1.9238
4	1.4998	1.4989	1.3999	1.1630	0.76444

TABLE 4 (*Continued*)

5	1.1629	1.0441	0.82247	0.48677	0.06084
6	0.86034	0.66443	0.38210	0.03295	−0.32004
7	0.59306	0.35396	0.05843	−0.24680	−0.47537
8	0.36249	0.10894	−0.16367	−0.38955	−0.47787
9	0.17129	−0.07282	−0.29711	−0.42722	−0.38634
10	0.02180	−0.19459	−0.35593	−0.39057	−0.25026
12	−0.15498	−0.28704	−0.31889	−0.21098	0.01050
14	−0.20395	−0.25165	−0.19309	−0.03416	0.14269
16	−0.18368	−0.17205	−0.07499	0.06593	0.15185
18	−0.13945	−0.09636	0.00146	0.09533	0.10267
20	−0.09483	−0.04108	0.03835	0.08428	0.04596

TABLE 5

Values of Modified Mathieu Function $Ce_4(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	1.1086	1.3389	1.7569	2.4173	3.4036
2	1.1402	1.3447	1.7094	2.2701	3.0767
3	1.1750	1.3539	1.6668	2.1337	2.7777
4	1.2090	1.3622	1.6246	2.0030	2.4995
5	1.2357	1.3633	1.5764	1.8712	2.2341
6	1.2485	1.3508	1.5161	1.7329	1.9764
7	1.2436	1.3212	1.4415	1.5868	1.7257
8	1.2209	1.2753	1.3539	1.4351	1.4843
9	1.1830	1.2159	1.2564	1.2812	1.2549
10	1.1330	1.1460	1.1523	1.1277	1.0391
12	1.0056	0.98400	0.93253	0.82970	0.65183
14	0.85343	0.80451	0.71006	0.55534	0.33119
16	0.69134	0.62346	0.50214	0.32192	0.08994
18	0.53665	0.45884	0.32677	0.14485	−0.06528
20	0.40354	0.32374	0.19407	- 0.02753	−0.14399
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	4.8340	6.8683	9.7122	13.6152	18.8542
2	4.1905	5.6763	7.5866	9.9321	12.6332
3	3.6170	4.6510	5.8388	7.0688	8.1249
4	3.1011	3.7660	4.4043	4.8617	4.9166
5	2.6316	2.9989	3.2303	3.1799	2.6894
6	2.2011	2.3346	2.2773	1.9215	1.1991
7	1.8078	1.7636	1.5147	1.0045	0.25483
8	1.4518	1.2788	0.91563	0.35942	−0.29441
9	1.1330	0.87224	0.45490	−0.07303	−0.56620
10	0.84996	0.53553	0.10988	−0.34146	−0.64899
12	0.38468	0.04324	−0.30499	−0.53216	−0.49268
14	0.04888	−0.24184	−0.44407	−0.44014	−0.17824
16	−0.15973	−0.35575	−0.40144	−0.23144	0.09096
18	−0.25421	−0.34593	−0.26929	−0.02886	0.22516
20	−0.26510	−0.26980	−0.12543	0.10202	0.23050

TABLE 6

Values of Modified Mathieu Function $Ce_5(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	1.1412	1.5282	2.2710	3.5374	5.6077
2	1.1563	1.5151	2.1935	3.3245	5.1202
3	1.1731	1.5042	2.1202	3.1239	4.6683

TABLE 6 (*Continued*)

4	1.1921	1.4957	2.0516	2.9353	4.2504
5	1.2132	1.4897	1.9873	2.7581	3.8641
6	1.2359	1.4855	1.9267	2.5910	3.5062
7	1.2590	1.4818	1.8680	2.4318	3.1731
8	1.2802	1.4763	1.8091	2.2778	2.8609
9	1.2970	1.4665	1.7472	2.1263	2.5659
10	1.3064	1.4497	1.6800	1.9752	2.2856
12	1.2955	1.3884	1.5250	1.6710	1.7661
14	1.2437	1.2906	1.3455	1.3692	1.3036
16	1.1582	1.1651	1.1512	1.0788	0.90042
18	1.0480	1.0205	0.95063	0.80629	0.55675
20	0.92029	0.86472	0.75164	0.55842	0.27427
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	8.9280	14.1878	22.4276	35.1766	54.6063
2	7.8948	12.0901	18.2886	27.1920	39.5195
3	6.9605	10.2507	14.7938	20.7544	28.0285
4	6.1173	8.6426	11.8561	15.6014	19.3790
5	5.3573	7.2398	9.3977	11.5090	12.9574
6	4.6719	6.0181	7.3491	8.2873	8.2676
7	4.0527	4.9552	5.6499	5.7767	4.9121
8	3.4918	4.0311	4.2479	3.8446	2.5754
9	2.9824	3.2293	3.0994	2.3820	1.0082
10	2.5195	2.5361	2.1678	1.2989	0.01418
12	1.7215	1.4339	0.83522	−0.01644	−0.84049
14	1.0801	0.65007	0.04279	−0.56769	−0.86016
16	0.57767	0.11871	−0.37198	−0.67437	−0.54741
18	0.19727	−0.21384	−0.52797	−0.54412	−0.17196
20	−0.07276	−0.38751	−0.51169	−0.31371	0.12777

TABLE 7

Values of Modified Mathieu Function $Se_1(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	.06840	.13532	.19914	.25799	.30961
2	.04709	.09159	.13086	.16208	.18239
3	.03295	.06296	.08715	.10274	.10727
4	.02347	.04404	.05895	.06582	.06293
5	.01700	.03131	.04047	.04256	.03661
6	.01251	.02260	.02815	.02771	.02090
7	.00934	.01654	.01981	.01811	.01152
8	.00705	.01224	.01408	.01184	.00593
9	.00538	.00916	.01009	.00772	.00264
10	.00415	.00691	.00728	.00499	.00075
12	.00253	.00404	.00385	.00197	−.00083
14	.00159	.00242	.00207	.00065	−.00111
16	.00102	.00149	.00111	.00009	−.00098
18	.00067	.00093	.00060	−.00012	−.00075
20	.00044	.00059	.00032	−.00018	−.00053
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	.35130	.37984	.39160	.38277	.34981
2	.18892	.17923	.15176	.10677	.04736
3	.09899	.07736	.04382	.00256	−.03898
4	.04974	.02742	−.00056	−.02823	−.04767
5	.02305	.00415	−.01572	−.03061	−.03443

TABLE 7 (*Continued*)

6	.00883	−.00566	−.01822	−.02392	−.01929
7	.00153	−.00889	−.01587	−.01590	−.00797
8	−.00197	−.00906	−.01215	−.00920	−.00104
9	−.00341	−.00795	−.00856	−.00440	.00249
10	−.00377	−.00644	−.00559	−.00130	.00381
12	−.00317	−.00366	−.00177	.00138	.00329
14	−.00219	−.00178	−.00004	.00169	.00172
16	−.00136	−.00070	.00054	.00123	.00053
18	−.00079	−.00015	.00061	.00069	−.00007
20	−.00042	.00008	.00049	.00030	−.00028

TABLE 8

Values of Modified Mathieu Function $Se_2(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	0.16828	0.33969	0.51700	0.70220	0.89595
2	0.13834	0.27606	0.41188	0.54311	0.66499
3	0.11228	0.22132	0.32313	0.41239	0.48190
4	0.09035	0.17576	0.25071	0.30867	0.34203
5	0.07234	0.13880	0.19312	0.22858	0.23834
6	0.05782	0.10933	0.14814	0.16794	0.16323
7	0.04622	0.08610	0.11343	0.12265	0.10981
8	0.03702	0.06788	0.08682	0.08914	0.07235
9	0.02973	0.05364	0.06650	0.06449	0.04641
10	0.02396	0.04251	0.05100	0.04642	0.02867
12	0.01573	0.02696	0.03013	0.02361	0.00874
14	0.01048	0.01733	0.01791	0.01154	0.00030
16	0.00709	0.01128	0.01068	0.00523	−0.00276
18	0.00486	0.00743	0.00638	0.00201	−0.00345
20	0.00337	0.00495	0.00380	0.00043	−0.00317
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	1.0969	1.3008	1.4992	1.6785	1.8187
2	0.76997	0.84717	0.88237	0.85901	0.76084
3	0.52245	0.52335	0.47414	0.36785	0.20632
4	0.34263	0.30332	0.22081	0.09992	−0.04168
5	0.21658	0.16068	0.07444	−0.02837	−0.12215
6	0.13090	0.07253	−0.00276	−0.07653	−0.12279
7	0.07428	0.02093	−0.03795	−0.08311	−0.09329
8	0.03790	−0.00711	−0.04926	−0.07112	−0.05861
9	0.01528	−0.02059	−0.04809	−0.05321	−0.02935
10	0.00180	−0.02546	−0.04128	−0.03567	−0.00851
12	−0.00948	−0.02312	−0.02421	−0.01017	0.01098
14	−0.01095	−0.01606	−0.01094	0.00207	0.01293
16	−0.00912	−0.00960	−0.00310	0.00583	0.00876
18	−0.00664	−0.00501	0.00069	0.00557	0.00415
20	−0.00446	−0.00214	0.00207	0.00396	0.00098

TABLE 9

Values of Modified Mathieu Function $Se_3(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	0.28177	0.58335	0.92520	1.3289	1.8174
2	0.25585	0.52460	0.81849	1.1481	1.5208
3	0.22907	0.46514	0.71366	0.97678	1.2509
4	0.20280	0.40767	0.61469	0.81981	1.0124

TABLE 9 (*Continued*)

5	0.17782	0.35376	0.52376	0.67953	0.80655
6	0.15465	0.30434	0.44202	0.55677	0.63263
7	0.13358	0.25990	0.36990	0.45135	0.48858
8	0.11473	0.22060	0.30732	0.36236	0.37149
9	0.09810	0.18633	0.25380	0.28836	0.27798
10	0.08362	0.15680	0.20856	0.22765	0.20449
12	0.06038	0.11023	0.13930	0.13881	0.10419
14	0.04348	0.07713	0.09212	0.08232	0.04723
16	0.03134	0.05394	0.06054	0.04734	0.01669
18	0.02267	0.03780	0.03961	0.02616	0.00146
20	0.01648	0.02658	0.02582	0.01359	−0.00527
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	2.4142	3.1416	4.0172	5.0475	6.2179
2	1.9378	2.3902	2.8537	3.2814	3.5986
3	1.5235	1.7698	1.9495	2.0066	1.8751
4	1.1728	1.2712	1.2683	1.1222	0.80428
5	0.88305	0.88107	0.77178	0.53732	0.18998
6	0.64900	0.58420	0.42370	0.17408	−0.12034
7	0.46430	0.36502	0.19076	−0.03205	−0.24102
8	0.32187	0.20848	0.04385	−0.13250	−0.25372
9	0.21456	0.10082	−0.04135	−0.16629	−0.21335
10	0.13560	0.03006	−0.08430	−0.16155	−0.15413
12	0.03941	−0.03848	−0.09867	−0.10630	−0.04642
14	−0.00455	−0.05363	−0.07407	−0.04790	0.01347
16	−0.02077	−0.04703	−0.04423	−0.00946	0.03233
18	−0.02360	−0.03432	−0.02094	0.00930	0.02931
20	−0.02082	−0.02216	−0.00612	0.01502	0.01890

TABLE 10

Values of Modified Mathieu Function $Se_4(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	0.39463	0.84503	1.4138	2.1776	3.2347
2	0.37627	0.79817	1.3148	1.9819	2.8633
3	0.35586	0.74793	1.2133	1.7897	2.5130
4	0.33385	0.69530	1.1107	1.6029	2.1858
5	0.31081	0.64145	1.0090	1.4239	1.8836
6	0.28730	0.58752	0.90979	1.2547	1.6076
7	0.26380	0.53448	0.81449	1.0967	1.3585
8	0.24073	0.48314	0.72419	0.95107	1.1363
9	0.21842	0.43413	0.63969	0.81833	0.94041
10	0.19712	0.38791	0.56153	0.69870	0.76968
12	0.15832	0.30511	0.42530	0.49800	0.49758
14	0.12520	0.23597	0.31565	0.34487	0.30507
16	0.09787	0.18015	0.23039	0.23236	0.17521
18	0.07590	0.13626	0.16591	0.15236	0.09165
20	0.05858	0.10243	0.11819	0.09707	0.04051
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	4.7138	6.7818	9.6490	13.5682	18.8187
2	4.0298	5.5534	7.4914	9.8579	12.5758
3	3.4110	4.4900	5.7143	6.9759	8.0609
4	2.8560	3.5782	4.2677	4.7731	4.8735
5	2.3634	2.8050	3.1063	3.1223	2.6921
6	1.9310	2.1575	2.1891	1.9148	1.2588

TABLE 10 (*Continued*)

7	1.5561	1.6228	1.4782	1.0577	0.36942
8	1.2350	1.1880	0.93945	0.47267	−0.13593
9	0.96346	0.84048	0.54207	0.09435	−0.38047
10	0.73712	0.56804	0.25870	−0.13131	−0.45716
12	0.40088	0.20320	−0.05883	−0.29067	−0.35966
14	0.18811	0.01168	−0.16636	−0.25361	−0.17308
16	0.06334	−0.07176	−0.16852	−0.15892	−0.02863
18	−0.00295	−0.09442	−0.12925	−0.07123	0.04656
20	−0.03308	−0.08730	−0.08258	−0.01164	0.06763

TABLE 11

Values of Modified Mathieu Function $Se_5(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	0.50806	1.1350	2.0261	3.3842	5.5114
2	0.49404	1.0935	1.9233	3.1498	5.0058
3	0.47893	1.0503	1.8205	2.9229	4.5311
4	0.46266	1.0055	1.7175	2.7032	4.0855
5	0.44522	0.95900	1.6143	2.4904	3.6675
6	0.42665	0.91094	1.5113	2.2847	3.2764
7	0.40710	0.86162	1.4089	2.0865	2.9116
8	0.38675	0.81145	1.3076	1.8965	2.5728
9	0.36580	0.76086	1.2082	1.7154	2.2599
10	0.34451	0.71035	1.1113	1.5437	1.9724
12	0.30177	0.61133	0.92761	1.2307	1.4718
14	0.26020	0.51760	0.76060	0.96037	1.0654
16	0.22112	0.43160	0.61301	0.73302	0.74519
18	0.18547	0.35491	0.48605	0.54707	0.50073
20	0.15383	0.28828	0.37963	0.39910	0.32040
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	8.8670	14.1488	22.4024	35.1601	54.5952
2	7.8187	12.0384	18.2527	27.1664	39.5007
3	6.8646	10.1820	14.7432	20.7166	28.0002
4	5.9972	8.5532	11.7889	15.5516	19.3442
5	5.2102	7.1291	9.3159	11.4525	12.9247
6	4.4984	5.8894	7.2595	8.2343	8.2493
7	3.8571	4.8161	5.5633	5.7400	4.9202
8	3.2821	3.8927	4.1773	3.8368	2.6181
9	2.7695	3.1041	3.0573	2.4121	1.0872
10	2.3154	2.4362	2.1638	1.3709	0.12471
12	1.5673	1.4110	0.92094	0.13225	−0.69719
14	1.0064	0.72162	0.21470	−0.37207	−0.73439
16	0.60145	0.28547	−0.13573	−0.47604	−0.48918
18	0.32205	0.03208	−0.26611	−0.39470	−0.21558
20	0.13950	−0.09651	−0.27382	−0.25559	−0.01570

TABLE 12

Values of Modified Mathieu Function $Se_6(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	0.62524	1.4688	2.8234	5.1539	9.2485
2	0.61333	1.4278	2.7067	4.8536	8.5252
3	0.60091	1.3863	2.5917	4.5649	7.8454
4	0.58792	1.3442	2.4785	4.2872	7.2067
5	0.57431	1.3015	2.3667	4.0198	6.6067

TABLE 12 (*Continued*)

6	0.56004	1.2581	2.2562	3.7624	6.0431
7	0.54506	1.2138	2.1469	3.5143	5.5139
8	0.52935	1.1687	2.0387	3.2753	5.0174
9	0.51290	1.1227	1.9317	3.0452	4.5517
10	0.49573	1.0760	1.8258	2.8237	4.1161
12	0.45945	0.98057	1.6184	2.4068	3.3294
14	0.42114	0.88388	1.4184	2.0256	2.6494
16	0.38169	0.78772	1.2283	1.6812	2.0693
18	0.34208	0.69405	1.0507	1.3748	1.5822
20	0.30326	0.60465	0.88754	1.1066	1.1806
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	16.4676	29.1545	51.3078	89.6491	155.255
2	14.7990	25.4199	43.1373	72.1129	118.287
3	13.2690	22.0897	36.0874	57.5680	89.0684
4	11.8672	19.1253	30.0221	45.5612	66.1482
5	10.5841	16.4914	24.8206	35.7016	48.3227
6	9.4108	14.1560	20.3757	27.6527	34.5956
7	8.3393	12.0899	16.5923	21.1260	24.1462
8	7.3621	10.2671	13.3865	15.8741	16.3000
9	6.4725	8.6636	10.6839	11.6855	10.5056
10	5.6645	7.2582	8.4189	8.3797	6.3134
12	4.2713	4.9650	4.9760	3.8237	1.3490
14	3.1428	3.2514	2.6685	1.2321	−0.73178
16	2.2438	2.0028	1.1913	−0.09221	−1.28226
18	1.5421	1.1221	0.30627	−0.63917	−1.12283
20	1.0078	0.52675	−0.17078	−0.74531	−0.72343

Numerical Quadrature over a Rectangular Domain in Two or More Dimensions

Part 2. Quadrature in several dimensions, using special points

By J. C. P. Miller

1. Introduction. In Part 1 [1], several approximate formulas were developed for quadrature over a rectangular domain (reduced to a square by scale changes) in the form

$$I = \int_{-rh}^{rh} \int_{-rh}^{rh} f(x, y)\, dx\, dy \doteq \sum A_{s,t}\, f(sh, th) \tag{1.1}$$

using some or all of a lattice of 9 or 16 points equally spaced over the square.

In this note, the restriction to equal spacing is relaxed, to allow points to be chosen to give greater accuracy in the approximation to the integral. The restriction to two dimensions or variables is also removed in some cases.

It is still assumed that the integrand $f(x_1, x_2, \cdots, x_n)$ may be expanded in a power series as far as we please in all variables—in other words, that for the precision desired, the integrand may be replaced by a polynomial of suitable degree. Considerations of symmetry once again ensure that we need concern ourselves only with terms of even degree in each variable separately. By scale changes we also make the range of integration $-h$ to $+h$ in each variable.

2. Expansion in Taylor Series. We need then, to find values $x_{s,t}$, A_t such that

$$I = \int_{-h}^{h} dx_1 \cdots \int_{-h}^{h} f(x_1, x_2, \cdots, x_n)\, dx_n \doteq (2h)^n \sum A_t f(x_{1,t}, x_{2,t}, \cdots, x_{n,t}) \tag{2.1}$$

where the points $\{x_{s,t}\}$ are chosen in symmetrical groups which are such that $\sum f(x_{1,t}, x_{2,t}, \cdots, x_{n,t})$ is free of all odd powers, just as the integral I is free of such powers by symmetry.

We expand $f(x_1, x_2, \cdots, x_n)$ as a Taylor series and evaluate the integral. We use the notation of Bickley [2], somewhat extended, namely,

$$S_2 = \sum_t \frac{\partial^2 \phi}{\partial x_t^2} \equiv \nabla^2 \phi \qquad S_{2,2} = \sum_{t,u}{}^{*} \frac{\partial^4 \phi}{\partial x_t^2 \partial x_u^2} \equiv \mathfrak{D}^4 \phi$$

$$S_{2,2,2} = \sum_{t,u,v}{}^{*} \frac{\partial^6 \phi}{\partial x_t^2 \partial x_u^2 \partial x_v^2} \equiv \mathfrak{I}^6 \phi \qquad S_{2,2,2,2} = \sum_{t,u,v,w}{}^{*} \frac{\partial^8 \phi}{\partial x_t^2 \partial x_u^2 \partial x_v^2 \partial x_w^2} \equiv \mathfrak{Q}^8 \phi \tag{2.2}$$

where the asterisk indicates that the suffixes t, u, v, w are unequal in pairs throughout.

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This gives, with f_0 for $f(0, 0, 0, \cdots, 0)$

$$J \equiv I/(2h)^n = f_0 + \frac{h^2}{3!}\nabla^2 f_0 + \frac{h^4}{5!}\left(\nabla^4 + \frac{4}{3}\mathfrak{D}^4\right) f_0$$

$$+ \frac{h^6}{7!}\left(\nabla^6 + 4\Delta^2\mathfrak{D}^4 + \frac{16}{3}\mathfrak{I}^6\right) f_0 \tag{2.3}$$

$$+ \frac{h^8}{9!}\left(\nabla^8 + 8\nabla^4\mathfrak{D}^4 + \frac{16}{5}\mathfrak{D}^8 + \frac{128}{5}\nabla^2\mathfrak{I}^6 + \frac{192}{5}\mathfrak{Q}^8\right) f_0 + \cdots$$

where we have used such formulas as

$$S_4 = \sum_i \frac{\partial^4\phi}{\partial x_i^4} = (\nabla^4 - 2\mathfrak{D}^4)\phi \qquad S_6 = \sum_i \frac{\partial^6\phi}{\partial x_i^6} = (\nabla^6 - 3\nabla^2\mathfrak{D}^4 + 3\mathfrak{I}^6)\phi$$

(2.4)

$$S_{4,2} = \sum_{\substack{i,u \\ i \neq u}} \frac{\partial^6\phi}{\partial x_i^4 \partial x_u^2} = (\nabla^2\mathfrak{D}^4 - 3\mathfrak{I}^6)\phi.$$

See David & Kendall [3] for extended tables of coefficients.

3. Summation over Sets of Points. We now choose appropriate sets of symmetrically placed points. We tabulate and label a few sets below.

(3.1)

Label	Coordinates	Number of Points
0	$(0, 0, \cdots, 0)$	1
$\alpha(a)$	$(\pm a, 0, \cdots, 0)$ with all permutations	$2n$
$\beta(b)$	$(\pm b, \pm b, 0, \cdots, 0)$ with all permutations	$2n(n-1)$
$\gamma(c, d)$	$(\pm c, \pm d, 0, \cdots, 0)$ with all permutations	$4n(n-1)$
$\epsilon(e)$	$(\pm e, \pm e, \pm e, 0, \cdots, 0)$ with all permutations	$4n(n-1)(n-2)/3$

In the formula (2.1) we have one arbitrary constant for the point 0, two constants for each set of type α, β, or ϵ, three constants for each set of type γ, and so on. We naturally wish to minimize the total number of points, but we note also that sets of type 0 and α alone are useless when dealing with terms involving $\mathfrak{D}^4$, $\mathfrak{I}^6$, $\mathfrak{Q}^8$, etc., while sets of 0, α, β, γ, only are useless for $\mathfrak{I}^6$ or $\mathfrak{Q}^8$, and so on.

If we expand $f(x_1, x_2, \cdots, x_n)$ at each point and sum over the set, we obtain the following expressions for the sums:

(3.21) 0 f_0

(3.22) $\alpha(a)$ $2nf_0 + \frac{2h^2}{2!}a^2\nabla^2 f_0 + \frac{2h^4}{4!}a^4 S_4 + \frac{2h^6}{6!}a^6 S_6 + \frac{2h^8}{8!}a^8 S_8 + \cdots$

$$= 2nf_0 + \frac{2h^2}{2!}a^2\nabla^2 f_0 + \frac{2h^4}{4!}a^4(\nabla^4 - 2\mathfrak{D}^4)f_0$$

$$+ \frac{2h^6}{6!}a^6(\nabla^6 - 3\nabla^2\mathfrak{D}^4 + 3\mathfrak{I}^6)f_0$$

$$+\frac{2h^5}{8!}a^5(\nabla^5-4\nabla^4\mathfrak{D}^4+2\mathfrak{D}^5+4\nabla^2\mathfrak{F}^6-4\mathfrak{Q}^8)f_0+\cdots$$

(3.23) $\beta(b)$ $$2n(n-1)f_0+\frac{4(n-1)h^2}{2!}b^2\nabla^2 f_0$$

$$+\frac{4h^4}{4!}b^4\{(n-1)\nabla^4-2(n-4)\mathfrak{D}^4\}f_0$$

$$+\frac{4h^6}{6!}b^6\{(n-1)\nabla^6-3(n-6)\nabla^2\mathfrak{D}^4+3(n-16)\mathfrak{F}^6\}f_0$$

$$+\frac{4h^8}{8!}b^8\{(n-1)\nabla^8-4(n-8)\nabla^4\mathfrak{D}^4+2(n+6)\mathfrak{D}^5$$
$$+4(n-43)\nabla^2\mathfrak{F}^6-4(n-64)\mathfrak{Q}^8\}f_0$$
$$+\cdots$$

(3.24) $\gamma(c,d)$ $$4n(n-1)f_0+\frac{4(n-1)h^2}{2!}(c^2+d^2)\nabla^2 f_0$$

$$+\frac{4h^4}{4!}[(n-1)(c^4+d^4)\nabla^4-2\{(n-1)(c^4+d^4)-6c^2d^2\}\mathfrak{D}^4]f_0$$

$$+\frac{4h^6}{6!}[(n-1)(c^6+d^6)\nabla^6$$
$$-3\{(n-1)(c^6+d^6)-5c^2d^2(c^2+d^2)\}\nabla^2\mathfrak{D}^4$$
$$+3\{(n-1)(c^6+d^6)-15c^2d^2(c^2+d^2)\}\mathfrak{F}^6]f_0$$

$$+\frac{4h^8}{8!}[(n-1)(c^8+d^8)\nabla^8$$
$$-4\{(n-1)(c^8+d^8)-7c^2d^2(c^4+d^4)\}\nabla^4\mathfrak{D}^4$$
$$+2\{(n-1)(c^8+d^8)-28c^2d^2(c^4+d^4)+70c^4d^4\}\mathfrak{D}^8$$
$$+4\{(n-1)(c^8+d^8)-7c^2d^2(c^4+d^4)-70c^4d^4\}\nabla^2\mathfrak{F}^6$$
$$-4\{(n-1)(c^8+d^8)-28c^2d^2(c^4+d^4)-70c^4d^4\}\mathfrak{Q}^8]f_0$$
$$+\cdots$$

(3.25) $\epsilon(e)$ $$\frac{4}{3}n(n-1)(n-2)f_0+\frac{4(n-1)(n-2)}{2!}h^2e^2\nabla^2 f_0$$

$$+\frac{4(n-2)}{4!}h^4e^4\{(n-1)\nabla^4-2(n-7)\mathfrak{D}^4\}f_0$$

$$+\frac{4h^6}{6!}e^6\{(n-1)(n-2)\nabla^6-3(n-2)(n-11)\nabla^2\mathfrak{D}^4$$
$$+3(n^2-33n+122)\mathfrak{F}^6\}f_0$$

$$+\frac{4h^8}{8!}e^{\delta}\{(n-1)(n-2)\nabla^8-4(n-2)(n-15)\nabla^4\mathfrak{D}^4$$
$$+2(n-2)(n+13)\mathfrak{D}^8+4(n^2-87n+380)\nabla^2\mathfrak{D}^6$$
$$-4(n^2-129n+1094)\mathfrak{D}^8\}f_0+\cdots$$

We use parts of (3.22), (3.23) and (3.25) in the present note; the rest is given for reference.

4. Formulas Accurate to Terms in h^4. We now seek a formula to give J accurately as far as terms in h^4, that is, including the first four distinct terms in (2.3). We need $\alpha(a)$ and $\beta(b)$ and shall retain the point 0; this gives five disposable constants, counting the three multipliers, which we shall now denote by A_0, A_α, A_β. This is apparently more than we need, but in fact, the fifth constant turns out to provide an essential extra degree of freedom.

We attempt, then, to satisfy as well as possible

$$J = A_0 f_0 + \sum A_\alpha f(x_\alpha) + \sum A_\beta f(x_\beta) \tag{4.1}$$

using x_α and x_β for typical *sets* of coordinates. If we now equate coefficients of f_0, $\nabla^2 f_0$, $\nabla^4 f_0$ and $\mathfrak{D}^4 f_0$, we obtain:

$$\begin{cases} A_0 + 2nA_\alpha + 2n(n-1)A_\beta = 1 \\ \qquad 2a^2A_\alpha + 4(n-1)b^2A_\beta = \frac{1}{3} \\ \qquad 2a^4A_\alpha + 4(n-1)b^4A_\beta = \frac{1}{5} \\ -4a^4A_\alpha - 8(n-4)b^4A_\beta = \frac{4}{15}. \end{cases} \tag{4.2}$$

We write $A_0 = \lambda$ and retain this, and solve for a, b, A_α, A_β. It may be noted that the second and third equations are no longer inconsistent, as they were for equally spaced points in Part 1 [1].

We use the last two equations to give A_α and A_β in terms of a^4 and b^4; then substitute in the first two equations and eliminate a^2 to give

$$\frac{1}{b^4}\frac{5n+4}{2}-\frac{30}{b^2}+\frac{18(5n-7)+9(14-5n)\lambda}{n(n-1)}=0 \tag{4.3}$$

whence

$$\frac{1}{b^2}=\frac{30}{5n+4}\pm\frac{6}{5n+4}\sqrt{\frac{(5n-14)\{(5n+4)\lambda-4\}}{2n(n-1)}}. \tag{4.4}$$

The values of $1/b^2$ are real for $\lambda = 0$ only if $n \leqq 2.8$, that is, if $n = 1$ or 2. When $n \geqq 3$, we must have $\lambda > 4/(5n+4)$ for real values of b^2, indicating that the fifth disposable constant is essential.

We next consider several special cases; error terms will be considered individually.

5. Quadrature over a Square. Case $n = 2$. We take $\lambda = 0$ in (4.3); then

$$\frac{7}{b^4}-\frac{30}{b^2}+27=0 \tag{5.1}$$

whence

$$\frac{1}{b^2} = 3 \qquad \frac{1}{a^2} = 0 \qquad A_\alpha = 0 \qquad A_\beta = \frac{1}{4} \tag{5.21}$$

or

$$\frac{1}{b^2} = \frac{9}{7} \qquad \frac{1}{a^2} = \frac{15}{7} \qquad A_\alpha = \frac{10}{49} \qquad A_\beta = \frac{9}{196}. \tag{5.22}$$

The first of these (5.21) is a degenerate case; it is in fact the product-Gauss two-point formula, with error term of order h^4 instead of the expected order h^6.

The second case (5.22) gives a useful approximate formula

$$\begin{aligned} J = I/(4h)^2 = \frac{10}{49}\Big\{ & f\left(\sqrt{\frac{7}{15}}\,h, 0\right) + f\left(-\sqrt{\frac{7}{15}}\,h, 0\right) \\ & + f\left(0, \sqrt{\frac{7}{15}}\,h\right) + f\left(0, -\sqrt{\frac{7}{15}}\,h\right)\Big\} \\ + \frac{9}{196}\Big\{ & f\left(\sqrt{\frac{7}{3}}\,h, \sqrt{\frac{7}{3}}\,h\right) + f\left(\sqrt{\frac{7}{3}}\,h, -\sqrt{\frac{7}{3}}\,h\right) \\ & + f\left(-\sqrt{\frac{7}{3}}\,h, \sqrt{\frac{7}{3}}\,h\right) + f\left(-\sqrt{\frac{7}{3}}\,h, -\sqrt{\frac{7}{3}}\,h\right)\Big\} \end{aligned} \tag{5.3}$$

with the main error term

$$\frac{h^6}{6!}\left(-\frac{212}{14175}\,\nabla^6 + \frac{1612}{4725}\,\nabla^2\mathfrak{D}^4\right) f_0. \tag{5.4}$$

We may also seek a formula with $a = b$; this turns out to be the "product-Gauss" formula, see §6. Another idea that comes to mind is to choose λ so that a and b are both rational; this does not seem to be possible, although one or other of a, b may take any value we please. Thus

(5.5) $a = 1$ gives $b^2 = \frac{5}{11}$ with $A_0 = \frac{64}{225}$ $A_\alpha = \frac{2}{45}$ $A_\beta = \frac{121}{900}$

(5.6) and main error term $-\frac{h^6}{6!}\left(\frac{4\nabla^6 + 268\nabla^2\mathfrak{D}^4}{1155}\right) f_0$;

(5.7) $b = 1$ gives $a^2 = \frac{2}{5}$ with $A_0 = -\frac{2}{9}$ $A_\alpha = \frac{5}{18}$ $A_\beta = \frac{1}{36}$

(5.8) and main error term $+\frac{h^6}{6!}\left(\frac{2\nabla^6 + 344\nabla^2\mathfrak{D}^4}{525}\right) f_0$.

Note the possibly useful opposition of signs in the two error terms.

6. Quadrature over Cube and Hypercube. We now consider general n.

Equation (4.4) gives the following restrictions for $A_0 = \lambda$ if a, b are to be real.

n	1	2	3	4	5
A_0	$\leqq \frac{4}{9}$	$\leqq \frac{2}{7}$	$\geqq \frac{4}{19}$	$\geqq \frac{1}{6}$	$\geqq \frac{4}{29}$.

(6.1)

Since we have one disposable constant, and $\lambda = 0$ is excluded for $n \geqq 3$, it is not as easy as in §5 to pick out an obviously convenient formula—there remains scope for investigation.

One choice is of interest. This is to put $a = b$. We then find

$$a^2 = b^2 = \frac{3}{5} \qquad A_0 = \frac{25n^2 - 115n + 162}{162}$$

(6.2)

$$A_\alpha = \frac{5(14 - 5n)}{162} \qquad A_\beta = \frac{25}{324}.$$

The points given by $a = \pm\sqrt{3/5}$, $b = \pm\sqrt{3/5}$ are the "Product-Gauss" points with at most two non-zero coordinates. In detail we have

(6.31) $n = 1$ $A_0 = \frac{4}{9}$ $A_\alpha = \frac{5}{18}$ $\left(A_\beta = \frac{25}{324}\right)$ 3 points

(6.32) $n = 2$ $A_0 = \frac{16}{81}$ $A_\alpha = \frac{10}{81}$ $A_\beta = \frac{25}{324}$ 9 points

(6.33) $n = 3$ $A_0 = \frac{7}{27}$ $A_\alpha = -\frac{5}{162}$ $A_\beta = \frac{25}{324}$ 19 points

(6.34) $n = 4$ $A_0 = \frac{17}{27}$ $A_\alpha = -\frac{5}{27}$ $A_\beta = \frac{25}{324}$ 33 points

We observe that for $n = 1, 2$ these are respectively the Gauss and Product-Gauss formulas. For $n \geqq 3$, although the error term is still of *order* h^6 in $I/(2h)^n$, the number of points is fewer than 3^n, (it is, in fact, $2n^2 + 1$) and the Gauss coefficients are lost.

The leading error term in $I/(2h)^n$ is

$$\frac{4h^6}{25.7!}\left(-\nabla^6 + 3\nabla^2\mathfrak{D}^4 - \frac{893}{6}\,\mathfrak{Z}^6\right) f_0 \tag{6.4}$$

which does not depend on n. Of course, $\mathfrak{Z}^6 f_0 = 0$ if $n = 1$ or 2.

For comparison we quote the 3-dimensional Product-Gauss formula, which uses the sets of points 0, $\alpha(\sqrt{3/5})$, $\beta(\sqrt{3/5})$ and $\epsilon(\sqrt{3/5})$.

(6.5) $n = 3$ $A_0 = \frac{64}{729}$ $A_\alpha = \frac{40}{729}$ $A_\beta = \frac{25}{729}$ $A_\gamma = \frac{125}{5832}$ 27 points

with leading error term

$$\frac{4h^6}{25.7!}\,(-\nabla^6 + 3\nabla^2\mathfrak{D}^4 - 3\mathfrak{Z}^6)\, f_0 \equiv -\frac{4h^6}{25.7!}\,S_6\,. \tag{6.6}$$

The form of error term for (6.33) and (6.5) suggests further the complete elimination of $\mathfrak{Z}^6 f_0$ to yield

(6.7) $n = 3$ $A_0 = \frac{430}{5103}$ $A_\alpha = \frac{289}{5103}$ $A_\beta = \frac{341}{10206}$ $A_\gamma = \frac{893}{40824}$

27 points

with leading error term

$$\frac{4h^6}{25.7!}\,(-\nabla^6 + 3\nabla^2\mathfrak{D}^4)\,f_0 \tag{6.8}$$

which vanishes for a harmonic function, and, in fact, also for the example of the next section with $f(x, y, z) = \cos x \cos y \cos z$, since this is a solution of the partial differential equation

$$\nabla^4\phi - 3\mathfrak{D}^4\phi = 0.$$

In this 3-dimensional case, the further relevant error terms are

$$\frac{4h^8}{1125.8!}\,(-11\nabla^8 - 16\nabla^4\mathfrak{D}^4 + 98\mathfrak{D}^8 + 70\nabla^2\mathfrak{D}^6)\,f_0\,. \tag{6.9}$$

7. Numerical Examples.

(i) As in Part 1 [1, §9] we use

$$J = \frac{1}{4}\,I = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} \cos x \cos y\, dxdy \doteqdot 0.70807\ 342$$

for a non-harmonic illustrative example for two-dimensional formulas, and list the integral (J), its error (E), and the estimated correction (C), which is minus the error estimate.

Formula (5.3) with $h = 1$ gives

$$J \doteqdot 0.707362 \quad \text{with} \quad E \doteqdot -0.000711 \quad \text{and} \quad C \doteqdot +0.000782,$$

and with $h = \frac{1}{2}$

$$J \doteqdot 0.70806\ 42 \quad \text{with} \quad E \doteqdot -0.00000\ 92 \quad \text{and} \quad C \doteqdot +0.00000\ 94.$$

The Gauss-product formula (6.32) with $h = 1$ gives

$$J \doteqdot 0.708125 \quad \text{with} \quad E \doteqdot +0.000052 \quad \text{and} \quad C \doteqdot -0.000064,$$

and with $h = \frac{1}{2}$

$$J \doteqdot 0.70807\ 415 \quad \text{with} \quad E \doteqdot +0.00000\ 073 \quad \text{and} \quad C \doteqdot -0.00000\ 076.$$

(ii) For an example with a harmonic integrand, as in Part 1 [1], we use

$$J' = \frac{1}{4}\,I = \frac{1}{4}\int_{0}^{1.2}\int_{0}^{1.2} \sin x \sinh y\, dxdy \doteqdot 0.12922\ 70591.$$

Formula (5.3) with $h = 1.2$ gives

$$J' \doteqdot 0.12922\ 71000 \quad \text{with} \quad E \doteqdot +0.00000\ 00409 \quad \text{and} \quad C \doteqdot -0.00000\ 00409,$$

while formula (6.32), the Gauss-product formula, also with $h = 1.2$, gives

$$J' \doteqdot 0.12922\ 70778 \quad \text{with} \quad E \doteqdot +0.00000\ 00187 \quad \text{and} \quad C \doteqdot -0.00000\ 00188.$$

In each of these two cases our original error estimate vanishes since the integrand is harmonic, and the error is of order h^8; in fact, they are respectively about $\frac{69128}{91125}\,h^8\mathfrak{D}^8 f_0/8!$ and $\frac{392}{1145}\,h^8\mathfrak{D}^8 f_0/8!$.

(iii) For a three-dimensional example, we consider

$$J = \frac{1}{8} I = \frac{1}{8} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \cos x \cos y \cos z \, dxdydz = \sin^3 1 \doteq 0.59582\ 3237.$$

With $h = 1$, formula (6.33) gives

$$J \doteq 0.59987 \quad \text{with} \quad E \doteq +0.00405 \quad \text{and} \quad C \doteq -0.00473$$

while formula (6.5), the Gauss-product formula, gives

$$J \doteq 0.595889 \quad \text{with} \quad E \doteq +0.000066 \quad \text{and} \quad C \doteq -0.000095$$

and the special formula (6.7) gives

$$J \doteq 0.595806 \quad \text{with} \quad E \doteq -0.000017 \quad \text{and} \quad C \doteq +0.000020$$

The last is exceptional in having an error of order h^8, since, as remarked in §6, $\sin x \sin y \sin z$ satisfies $\nabla^4\phi - 3\mathfrak{D}^4\phi = 0$. It exhibits accidentally what may be expected with a harmonic integrand.

With $h = \frac{1}{2}$ these three formulas give

(6.33) $J \doteq 0.595871$ with $E \doteq +0.000048$ and $C \doteq -0.000050$

(6.5) $J \doteq 0.59582\ 415$ with $E \doteq +0.00000\ 091$ and $C \doteq -0.00000\ 100$

(6.7) $J \doteq 0.59582\ 319$ with $E \doteq -0.00000\ 004$ and $C \doteq +0.00000\ 0054$

(iv) For $n = 4$, we consider

$$J = \frac{1}{16} I = \frac{1}{16} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \cos x \cos y \cos z \cos w \, dxdydzdw$$

$$= \sin^4 1 \doteq 0.50136\ 80.$$

With $h = 1$, we have

(6.34) $J \doteq 0.514$ with $E \doteq +0.013$ and $C \doteq -0.019$ 33 points

$(\text{Gauss})^4$ $J \doteq 0.501441$ with $E \doteq +0.000073$ 81 points

and with $h = \frac{1}{2}$, we have

(6.34) $J \doteq 0.50153$ with $E \doteq +0.00016$ and $C \doteq -0.00017$

$(\text{Gauss})^4$ $J \doteq 0.50136\ 90$ with $E \doteq +0.00000\ 10$.

These numerical examples illustrate that the extra points used in the Gauss-product formulas may, sometimes at any rate, have a useful effect in reducing error, even though the order of the error, as represented by the power of h in its leading term, remains the same. Comparison of (6.4) and (6.6) shows a much reduced coefficient of $3^6 f_0$ in the latter, and (6.7) is a definite improvement on the Gauss-product in the present case. More investigation is clearly needed of other formulas and of other integrands, particularly those which do not so easily separate into a product of integrals.

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On the Propagation of Round-Off Errors in the Numerical Integration of the Heat Equation

By Arnold N. Lowan

1. Preliminary Considerations. For the sake of convenience we begin with a summary of facts pertaining to M-dimensional vectors. These facts will be needed in the subsequent developments:

A set of M real quantities $u_1, u_2, \cdots, u_M$ are said to represent an M-dimensional vector, to be denoted by $\mathbf{u}$. The u_h's will be referred to as the components of the vector $\mathbf{u}$.

For two M-dimensional vectors $\mathbf{u}$ and $\mathbf{v}$, the scalar product $(\mathbf{u}, \mathbf{v})$ will be defined by

$$(\mathbf{u}, \mathbf{v}) = \frac{1}{M}(u_1v_1 + u_2v_2 + \cdots u_Mv_M) = \frac{1}{M}\sum_{h=1}^{M} u_hv_h. \tag{1}$$

If $(\mathbf{u}, \mathbf{v}) = 0$ the vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be orthogonal. In particular

$$(\mathbf{u}, \mathbf{u}) = \frac{1}{M}\sum_{h=1}^{M} u_h^2. \tag{2}$$

The norm of the vector $\mathbf{u}$ to be denoted by $\| \mathbf{u} \|$ is defined by

$$\| \mathbf{u} \| = \left\{\frac{1}{M}\sum_{h=1}^{M} u_h^2\right\}^{1/2}. \tag{3}$$

If

$$\mathbf{u} = \sum \mathbf{u}_p \tag{4}$$

it can be readily shown that

$$\| \mathbf{u} \| \leqq \sum \| \mathbf{u}_p \|. \tag{5}$$

This is the Minkowski inequality.

Let now $\mathbf{v} = A\mathbf{u}$ where A is a symmetric square matrix of order M. We shall prove that if the eigenvalues of the matrix A are known to be numerically smaller than unity, then

$$\| \mathbf{v} \| \leqq \| \mathbf{u} \|.$$

Indeed let

$$\mathbf{u} = \sum_{r=1}^{M} \alpha_r \mathbf{w}_r \tag{6}$$

where the $\mathbf{w}_r$'s are the eigenvectors corresponding to the eigenvalues λ_r. From

$$A\mathbf{w}_r = \lambda_r \mathbf{w}_r \tag{7}$$

it is readily seen that

$$\| \mathbf{u} \| = \sqrt{(\mathbf{u}, \mathbf{u})} = \sqrt{\sum_{r=1}^{M} \alpha_r^2} \tag{8}$$

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provided that the $\mathbf{w}_r$'s are normalized so that

$$(\mathbf{w}_r, \mathbf{w}_r) = 1 \tag{9}$$

Clearly then

$$\mathbf{v} = A\mathbf{u} = \sum_{r=1}^{M} \alpha_r A\mathbf{w}_r = \sum_{r=1}^{M} \alpha_r \lambda_r \mathbf{w}_r$$

whence

$$\| \mathbf{v} \| \leqq \sqrt{\sum_{r=1}^{M} \alpha_r^2}$$

and therefore

$$\| A\mathbf{u} \| \leqq \| \mathbf{u} \| \tag{10}$$

since the λ_r's were assumed to be numerically smaller than unity.

The inequality (10) may be generalized in two important ways. Clearly

$$\| A^2\mathbf{u} \| \leqq \| A\mathbf{u} \|$$

and therefore

$$\| A^2\mathbf{u} \| \leqq \| \mathbf{u} \|$$

and more generally

$$\| A^n\mathbf{u} \| \leqq \| \mathbf{u} \| \tag{11}$$

provided the eigenvalues of A are numerically smaller than unity. Consider now

$$\mathbf{v}_n = A_n A_{n-1} \cdots A_2 A_1 \mathbf{u}$$

where the A_p's are symmetric matrices whose eigenvalues are numerically smaller than unity. We have in succession

$$\| \mathbf{v}_1 \| = \| A_1\mathbf{u} \| \leqq \| \mathbf{u} \|$$

$$\| \mathbf{v}_2 \| = \| A_2\mathbf{v}_1 \| \leqq \| \mathbf{v}_1 \| \leqq \| \mathbf{u} \|$$

$$\vdots$$

so that ultimately

$$\| A_n A_{n-1} \cdots A_2 A_1 \mathbf{u} \| \leqq \| \mathbf{u} \|. \tag{12}$$

It should be pointed out that the inequalities (11) and (12) are equally valid for nondefective matrices* whose eigenvalues are numerically smaller than unity.

Consider now the explicit difference analog

$$T_{h,k+1} = (1 - 2r)T_{h,k} + r(T_{h-1,k} + T_{h+1,k}) \tag{13}$$

of the differential equation

$$\rho c \frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2} \tag{13*}$$

---

* A square matrix of order M is nondefective when it has M distinct eigenvectors.

where $T_{h,k} = T(h\Delta x, k\Delta t)$ and $r = (K\Delta t)/(\rho c(\Delta x)^2$ where K, ρ and c are assumed constant. For the sake of concreteness assume that we are dealing with the problem of heat conduction in a slab whose bounding planes $x = 0$ and $x = a$ are kept at 0°C. Equation (13) may then be written in the compact form

$$\mathbf{T}_{k+1} = A\mathbf{T}_k \tag{14}$$

where $\mathbf{T}_k$ and $\mathbf{T}_{k+1}$ are the M-dimensional vectors whose components are the temperatures at times $k\Delta t$ and $(k+1)\Delta t$ at the mesh-points $h\Delta x$, $h = 1, 2, 3, \cdots, M$ where $(M+1)\Delta x = a$ and A is the tridiagonal $M \times M$ matrix whose elements on the principal diagonal are $= 1 - 2r$ and whose elements off the principal diagonal are $= +r$.

Starting with the initial temperature vector $\mathbf{T}_0$ equation (14) yields in succession

$$\begin{aligned} &\text{I} && \mathbf{T}_1 = A\mathbf{T}_0 \\ &\text{II} && \mathbf{T}_2 = A\mathbf{T}_1 \\ &\vdots && \vdots \\ & && \mathbf{T}_n = A\mathbf{T}_{n-1} \end{aligned} \tag{15}$$

If the computations involved in the successive steps of (15) could be carried out to an infinite number of decimal places the vectors $\mathbf{T}_k$ thus generated would be the true solutions of the difference equation (14). In actual practice the computations are carried to some fixed number of decimal places and the question arises: what is the error propagated as a result of rounding-off the values of the products in (13) at the various steps in the process of computation?

For the sake of concreteness assume that the initial temperatures are exact and that the computations are carried to p decimal places. Then, since formula (13) involves two multiplications, each one of which involves a round-off error ranging from $-\frac{1}{2} \times 10^{-p}$ to $\frac{1}{2} \times 10^{-p}$, it is clear that the first step in the sequence of operations (15) does not yield the true vector $\mathbf{T}_1 = A\mathbf{T}_0$ but the approximate vector $\mathbf{T}_1^* = A\mathbf{T}_0 + \boldsymbol{\delta}_1$ where $\boldsymbol{\delta}_1$ is the vector whose components represent the sum of the round-off errors corresponding to the two multiplications in (13). In entirely similar manner it is seen that the second step in the sequence of operations (15) yields the vector $\mathbf{T}_2^* = A\mathbf{T}_1^* + \boldsymbol{\delta}_2 = A(A\mathbf{T}_0 + \boldsymbol{\delta}_1) + \boldsymbol{\delta}_2 = A^2\mathbf{T}_0 + A\boldsymbol{\delta}_1 + \boldsymbol{\delta}_2$. Proceeding in this manner it is readily seen that when n successive steps of (15) have been carried out, we have generated the vector

$$\mathbf{T}_n^* = A^n\mathbf{T}_0 + A^{n-1}\boldsymbol{\delta}_1 + A^{n-2}\boldsymbol{\delta}_2 + \cdots + A\boldsymbol{\delta}_{n-1} + \boldsymbol{\delta}_n \tag{16}$$

where in general $\boldsymbol{\delta}_p$ is the error vector whose components represent the sum of the round-off errors in the arithmetical operations leading from the components of $\mathbf{T}_{p-1}$ to those of $\mathbf{T}_p$.

Clearly $\mathbf{T}_n^* - A^n\mathbf{T}_0 = \mathbf{E}_n$ is the round-off error vector corresponding to the nth time step. Thus

$$\mathbf{E}_n = A^{n-1}\boldsymbol{\delta}_1 + A^{n-2}\boldsymbol{\delta}_2 + \cdots A\boldsymbol{\delta}_{n-1} + \boldsymbol{\delta}_n . \tag{17}$$

In view of (5) and (11) the last equation yields

$$\| \mathbf{E}_n \| \leq \| \boldsymbol{\delta}_1 \| + \| \boldsymbol{\delta}_2 \| + \cdots + \| \boldsymbol{\delta}_{n-1} \| + \| \boldsymbol{\delta}_n \| \tag{18}$$

since the eigenvalues of A are known to be numerically smaller than unity.

Let δ^* denote an upper bound of the components of all vectors $\boldsymbol{\delta}_q$. It is then readily seen that

$$\| \boldsymbol{\delta}_q \| \leqq \delta^* \tag{19}$$

whence

$$\| \mathbf{E}_n \| \leqq n\delta^*. \tag{20}$$

Since

$$\| \mathbf{E}_n \| = \left\{ \frac{1}{M} \sum_{h=1}^{M} E_{nh}^2 \right\}^{1/2}$$

it is clear that the maximum value of any of its components E_{nh} is obtained by assuming that all but one of the components are equal to zero. Calling the maximum value of the component E_n^* the last inequality yields

$$E_n^* \leqq n\sqrt{M}\delta^*. \tag{21}$$

The second member of (21) is an upper bound of the round-off errors in the values of the temperatures generated by the explicit scheme (13). To illustrate, assume that $M = 49$ and $n = 100$. Since (13) involves two multiplications so that $\delta^* = 2 \cdot \frac{1}{2} \times 10^{-p} = 10^{-p}$ it follows that $E_{100}^* \leqq 100\sqrt{49} \times 10^{-p} = 7 \times 10^{-(p-2)}$. Thus on the basis of (21) the values of the temperatures for $t = 100\Delta t$ computed by the difference scheme (13) may be incorrect by not more than 7 units in the $(p-2)$th place.

For the explicit scheme under consideration, a somewhat lower upper bound than that given by (21) may be obtained as follows:

If $E_{h,k}$ denotes the absolute value of the error in $T_{h,k}$ and E_k^* denotes the largest of the values of $E_{h,k}$ (for $h = 1, 2, 3, \cdots M$) then, since $r \leqq \frac{1}{2}$ and therefore $1 - 2r \geqq 0$, the difference equation (13) yields:

$$\begin{aligned} E_{h,k+1} &\leqq (1-2r)E_k^* + r(E_k^* + E_k^*) + 2 \cdot \tfrac{1}{2} \times 10^{-p} \\ &= E_k^* + 10^{-p} = E_k^* + \delta^*. \end{aligned} \tag{22}$$

Since we assumed that the initial temperatures are exact so that $E_0^* = 0$, the last inequality yields

$$E_n^* \leqq n\delta^*. \tag{21*}$$

Thus the above elementary analysis has yielded a lower upper bound of the round-off errors than the previous more elegant analysis. It should be pointed out, however, that the virtue of the analysis which culminated in (21) lies in the fact that $n\sqrt{M}\delta^*$ is also the upper bound of the round-off errors in the implicit difference scheme

$$T_{h,k+1} = T_{h,k} + \frac{r}{2}(T_{h-1,k+1} - 2T_{h,k+1} + T_{h+1,k+1} + T_{h-1,k} - 2T_{h,k} + T_{h+1,k}) \tag{23}$$

$$h = 1, 2, 3, \cdots M$$

Indeed (23) may be written in the form

$$A\mathbf{T}_{k+1} = B\mathbf{T}_k = (4I - A)\mathbf{T}_k \tag{23*}$$

whence

$$\mathbf{T}_{k+1} = A^{-1}B\mathbf{T}_k = (4A^{-1} - I)\mathbf{T}_k = C\mathbf{T}_k \quad \text{(say)} \tag{24}$$

where A is the $M \times M$ tridiagonal matrix whose elements on the principal diagonal are $= 2 + 2r$ and whose elements off the principal diagonal are $= -r$ and where I is the $M \times M$ unit matrix. Since the determination of C involves the inversion of the matrix A, it may be easily shown that the counterpart of (16) is

$$T_n^* = C^n T_0 + C^{n-1}\bar{\delta}_1 + C^{n-2}\bar{\delta}_2 + \cdots + C\bar{\delta}_{n-1} + \bar{\delta}_n \tag{16*}$$

where now δ_0 is the error vector whose components represent the aggregate of the errors arising both from the replacement of $C = 4A^{-1} - I$ by $C^* = 4A^{-1*} - I$ where A^{-1*} is an inexact inverse of A and from the rounding-off of all products and quotients involved, to the number of places carried in the computation. Since the eigenvalues of C are numerically smaller than unity (see for instance, the writer's monograph on "The operator approach to stability and convergence") the developments which previously led from (16) to (20) and (21) apply with the sole exception that now δ^* refers to the vectors in (16*). It should be clear of course that in the present case the value of δ^* depends on the particular scheme for solving the system of equations (23) for the unknown temperatures $T_{h,k+1}$, or, what amounts to the same thing, the particular scheme for inverting the matrix A in (23*) To illustrate, we shall derive the expression for δ^* for the case where the system (23) is solved by the method of iteration. We shall also quote the results of an earlier RAD* report dealing with the analysis of errors for a different method of solution of the system (23).

2. The Method of Iteration. If $T_{h,k}^{(q)}$ denotes the qth approximation to the solution of (23), then the $(q + 1)$st approximation is given by

$$T_{h,k+1}^{(q+1)} = T_{h,k} + \frac{r}{2}\left(T_{h-1,k+1}^{(q)} - 2T_{h,k+1}^{(q+1)} + T_{h+1,k+1}^{(q)} + T_{h-1,k} - 2T_{h,k} + T_{h+1,k}\right)$$

whence

$$T_{h,k+1}^{(q+1)} = \frac{r}{2(1+r)}\left(T_{h-1,k+1}^{(q)} + T_{h+1,k+1}^{(q)}\right) + \frac{r}{2(1+r)}\left(T_{h-1,k} + T_{h+1,k}\right) + \frac{1-r}{1+r}T_{h,k}. \tag{25}$$

As before let E_k^* denote the largest of the absolute errors in the values of $T_{h,k}$ and let $\alpha^{(q)}$ denote the largest of the absolute errors in the qth approximation to the $T_{h,k+1}$'s. Then the last equation yields

$$E(T_{h,k+1}^{(q+1)}) \leqq \frac{r}{2(1+r)} \cdot 2\alpha^{(q)} + \frac{r}{2(1+r)} \cdot 2E_k^* + \frac{|1-r|}{1+r}E_k^* + 3\delta \tag{26}$$

where $3\delta = 3 \cdot \frac{1}{2} \times 10^{-p}$ is the sum of the absolute values of the maximum round-off errors corresponding to the three multiplications in (25).

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* AVCO Advanced Research and Development Division

For $r < 1$, (26) yields

$$E(T_{h,k+1}^{(q+1)}) \leqq \frac{r}{1+r}\alpha^{(q)} + \frac{1}{1+r}E_k^* + 3\delta = \rho\alpha^{(q)} + \sigma E_k^* + 3\delta \quad \text{(say)}$$

whence

$$\alpha^{(q+1)} \leqq \rho\alpha^{(q)} + \sigma E_k^* + 3\delta \tag{27}$$

where $\alpha^{(q+1)}$ is the largest of the absolute errors in the $(q+1)$th approximation to the $T_{h,k+1}$'s. Applying the last inequality to $p = 1, 2, 3, \cdots N-1$ we get

$$\alpha^{(2)} \leqq \rho\alpha^{(1)} + \sigma E_k^* + 3\delta$$

$$\alpha^{(3)} \leqq \rho\alpha^{(2)} + \sigma E_k^* + 3\delta$$

$$\vdots$$

$$\alpha^{(N)} \leqq \rho\alpha^{(N-1)} + \sigma E_k^* + 3\delta$$

whence

$$\alpha^{(N)} \leqq (1 + \rho + \rho^2 + \cdots + \rho^{N-2})(\sigma E_k^* + 3\delta) + \rho^{N-1}\cdot\alpha^{(1)}$$

$$\cong \frac{\sigma}{1-\rho}E_k^* + \frac{3\delta}{1-\rho} = \frac{\frac{1}{1+r}}{1-\frac{r}{1+r}}E_k^* + \frac{3\delta}{1-\frac{r}{1+r}} = E_k^* + 3(1+r)\delta.$$

In the last inequality N is the minimum number of iterations such that successive values of $T_{h,k+1}^{(p)}$ and $T_{h,k+1}^{(p+1)}$ agree to within a preassigned tolerance. Clearly $\alpha^{(N)}$ represents the maximum absolute round-off errors in the values of $T_{h,k+1}$. We have thus reached the conclusion

$$E_{k+1}^* \leqq E_k^* + 3(1+r)\delta. \tag{28}$$

From (28) it follows that for the method of iteration under consideration $\delta^* = 3(1+r)\delta$.

In the above analysis we assumed that $r < 1$. If $r \geqq 1$, $\sigma = (2r-1)/(1+r)$ and the counterpart of (28) is

$$E_{k+1}^* \leqq (2r-1)E_k^* + 3(1+r)\delta. \tag{28*}$$

Comparison between (28) and (28*) shows clearly that although the choice $r > 1$ seems to imply larger errors, the expression for δ^* is formally the same as for the case $r \leqq 1$, namely, $3(1+r)\delta$.

We now turn to an alternative method for the solution of the system. The method, of unknown origin, consists of the following algorithm: if (23*) is rewritten in the form

$$\mathbf{Ay} = \mathbf{b} \tag{29}$$

then the components of $\mathbf{y}$ are given by the following sequence of operations

$$
\begin{aligned}
&\beta_k = 2 + 2r - \frac{r^2}{\beta_{k-1}} && k = 1, 2, 3, \cdots M;\ \beta_1 = 2 + 2r \\
&\gamma_k = -\frac{r}{\beta_k} && \\
&z_k = \frac{1}{\beta^k}\,(b_k + rz_{k-1}) && z_1 = \frac{b_1}{2 + 2r} \\
&y_k = z_k - \gamma_k z_{k+1} && y_M = z_M .
\end{aligned} \tag{30}
$$

The analysis of the errors involved in the above algorithm is given in the writer's RAD report entitled "On the Propagation of the Errors in the Inversion of Certain Tridiagonal Matrices." The conclusion reached in this report is that if the components of $\mathbf{b}$ are assumed exact, an upper bound of the errors in the values of the components of $\mathbf{y}$ is given by the inequality

$$
E^*(y) \leqq \frac{\beta_*}{\beta_* - r}\left\{\frac{\beta_*}{\beta_* - r}\left[1 + \frac{B + rZ}{\beta_*^2 - r^2}\right] + Y\left(1 + \frac{r}{\beta_*^2 - r^2}\right) + 1\right\}\delta \tag{31}
$$

where $\beta_* = 1 + r + \sqrt{1 + 2r}$ and B, Z and Y are the largest absolute values of the b_k's, z_k's and y_k's and $\delta = \frac{1}{2} \times 10^{-p}$. A somewhat larger upper bound is obtained if Z and Y are replaced by upperbounds of the $|\, z_k \,|$'s and $|\, y_k \,|$'s which may be easily obtained from the above algorithm. We are led to

$$
\begin{aligned}
E^*(y) \leqq \frac{\beta_*}{\beta_* - r}\Bigg\{\frac{\beta_*}{\beta_* - r}\left[1 + \frac{B\beta_*}{(\beta_* - r)(\beta_*^2 - r^2)}\right] \\
+ \frac{B\beta_*}{(\beta_* - r)^2}\left(1 + \frac{r}{\beta_*^2 - r^2}\right) + 1\Bigg\}\delta.
\end{aligned} \tag{31*}
$$

Either one of the second members of the above inequalities plays the role of the quantity δ^* in (21). It will be noted that since $\mathbf{b}$ is the vector $B\mathbf{T}_k$ so that

$$
b_{h,k} = (2 - 2r)T_{h,k} + r(T_{h-1,k} + T_{h+1,k})
$$

it follows that

$$
\begin{aligned}
&b_k^* \leqq (2 - 2r)T_k^* + 2rT_k^* = 2T_k^* && \text{for } r \leqq 1 \\
&b_k^* \leqq (2r - 2)T_k^* + 2rT_k^* = (4r - 2)T_k^* && \text{for } r \geqq 1
\end{aligned}
$$

where b_k^* and T_k^* are upper bounds of $b_{h,k}$ and $T_{h,k}$ for fixed k respectively. If b^* and T^* denote upper bounds of $b_{h,k}$ and $T_{h,k}$ for all values of k, then

$$
b^* \leqq 2T^* \qquad \text{for } r \leqq 1 \tag{32}
$$

$$
b^* \geqq (4r - 2)T^* \qquad \text{for } r \geqq 1. \tag{32*}
$$

Since the temperature was assumed to vanish for $x = 0$ and $x = a$, it is clear that T^* is merely the maximum of the initial temperature function $f(x)$.

The previous developments are based on the assumption that the temperature vanishes on the boundaries of the slab. We shall now briefly discuss the modifications which must be made for other types of boundary conditions.

Assume first that the boundary conditions are

$$\begin{cases} T(0, t) = \phi_0(t) \\ T(a, t) = \phi_1(t). \end{cases} \tag{33}$$

If in the difference equation (13) we put $h = 1$ and $h = M$, we get

$$T_{1,k+1} = rT_{0,k} + (1 - 2r)T_{1,k} + rT_{2,k} = r\phi_0(k\Delta t) + (1 - 2r)T_{1,k} + rT_{2,k} \tag{34}$$

$$\begin{aligned} T_{M,k+1} &= rT_{M-1,k} + (1 - 2r)T_{M,k} + rT_{M+1,k} \\ &= rT_{M-1,k} + (1 - 2r)T_{M,k} + r\phi_1(k\Delta t). \end{aligned} \tag{35}$$

In view of (34) and (35) it can be readily seen that the counterpart of the matrix-vector equation (14) is

$$\mathbf{T}_{k+1} = A\mathbf{T}_k + \mathbf{u}_k \tag{36}$$

where

$$u_{1,k} = r\phi_0(k\Delta t), \tag{37}$$

$$u_{M,k} = r\phi_1(k\Delta t), \tag{38}$$

and all other components of $\mathbf{u}$ are zero.

If in (36) we put $k = 0, 1, 2, \cdots, n - 1$ we ultimately get

$$\mathbf{T}_n = A^nT_0 + A^{n-1}\mathbf{u}_0 + A^{n-2}\mathbf{u}_1 + \cdots A\mathbf{u}_{n-2} + \mathbf{u}_{n-1}. \tag{39}$$

The analysis of propagation of errors is entirely similar to the analysis which led to equation (16). Its counterpart is

$$\begin{aligned} \mathbf{T}^* = A^n\mathbf{T}_0 + A^{n-1}\mathbf{u}_0 + A^{n-2}\mathbf{u}_1 + \cdots + A\mathbf{u}_{n-2} + \mathbf{u}_{n-1} \\ + A^{n-1}\boldsymbol{\delta}_1 + A^{n-2}\boldsymbol{\delta}_2 + \cdots + A\boldsymbol{\delta}_{n-1} + \boldsymbol{\delta}_n \end{aligned} \tag{16*}$$

whence, in view of (39)

$$\mathbf{E}_n = A^{n-1}\boldsymbol{\delta}_1 + A^{n-2}\boldsymbol{\delta}_2 + \cdots + A\boldsymbol{\delta}_{n-1} + \boldsymbol{\delta}_n. \tag{16}$$

Thus, the expression of the error vector $\mathbf{E}_n$ is identical with that previously derived for the case where the temperature vanishes on the boundaries of the slab. We conclude that the upper bound of the round-off error is once more given by (21). The same conclusion can obviously be drawn also for the implicit difference scheme (23); the reasoning is identical with that given before. It should be pointed out, however, that the quantity T^* in equations (32) is the largest of the maxima of the functions $f(x)$, $\phi_0(t)$ and $\phi_1(t)$.

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The Rational Approximation of Functions which Are Formally Defined by a Power Series Expansion

By P. Wynn

1. Introduction. The advent of high speed digital computers and the consequent intensification of interest in the study of numerical analysis has caused considerable attention to be paid to the problem of obtaining approximation formulas for functions which occur in the theory of mathematical physics. It is the purpose of this note to describe the theory underlying various methods of obtaining rational approximations to functions which are formally defined by a power series expansion; it is assumed that the power series concerned are quite general in character, and that the functions with which they are associated do not satisfy a particular functional equation which would permit the use of any special method. The theory is then subjected to a detailed analysis in terms of the computational steps involved, and a comparison, with regard to computational efficiency, of the various methods which may be devised for obtaining rational approximations is given.

2. The Padé Table and the E-Array. The approximation to the function defined by the power series

$$\beta(x) = c_0 + c_1 x + c_2 x^2 + \cdots \tag{2.1}$$

by a rational function of the form

$$\frac{U_{\mu,\nu}(x)}{V_{\mu,\nu}(x)} = \frac{\alpha_{\mu,0} + \alpha_{\mu,1} x + \cdots + \alpha_{\mu,\nu} x^{\nu}}{\beta_{0,\nu} + \beta_{1,\nu} x + \cdots + \beta_{\mu,\nu} x^{\mu}} \tag{2.2}$$

can be systematized [1], [2, p. 420], [3, 377] by imposing the condition, where this is possible, that the expansion of (2.2) in ascending powers of x should agree with the power series (2.1) as far as the term in $x^{\mu+\nu}$, i.e., that the power series expansions of $\beta(x)V_{\mu,\nu}(x) - U_{\mu,\nu}(x)$ should commence with a term in $x^{\mu+\nu+1}$. This leads immediately to the set of linear equations.

$$\begin{aligned}
&c_0\beta_{0,\nu} = \alpha_{\mu,0}\\
&c_1\beta_{0,\nu} + c_0\beta_{1,\nu} = \alpha_{\mu,1}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&c_\nu\beta_{0,\nu} + c_{\nu-1}\beta_{1,\nu} + \cdots + c_{\nu-\mu}\beta_{\mu,\nu} = \alpha_{\mu,\nu}\\
&c_{\nu+1}\beta_{0,\nu} + c_\nu\beta_{1,\nu} + \cdots + c_{\nu-\mu+1}\beta_{\mu,\nu} = 0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&c_{\nu+\mu}\beta_{0,\nu} + c_{\nu+\mu-1}\beta_{1,\nu} + \cdots + c_\nu\beta_{\mu,\nu} = 0
\end{aligned} \tag{2.3}$$

which serves uniquely to define the coefficients in (2.2) if the restriction $\beta_{0,\nu} = 1$ is imposed. The rational functions (2.2), determined by equations (2.3), may be placed in an array in which the quotient $U_{\mu,\nu}(x)/V_{\mu,\nu}(x)$ is at the intersection of

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the $(\nu+1)$th column and the $(\mu+1)$th row. Such an array is referred to as the Padé table of the function $\beta(x)$, and the condition that it may be constructed from the systems of equations (2.3) is that all the Hankel determinants

$$H_m^{(n)} = \begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+n-1} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+n} \\ \vdots & \vdots & & \vdots \\ c_{m+n-1} & c_{m+n} & \cdots & c_{m+2n-2} \end{vmatrix} \tag{2.4}$$

should be non-zero.

Since the successive convergents of the continued fraction

$$\frac{c_m}{1-}\ \frac{q_1^{(m)}x}{1-}\ \frac{e_1^{(m)}x}{1-}\ \cdots\ \frac{q_r^{(m)}x}{1-}\ \frac{e_r^{(m)}x}{1-}\ \cdots \tag{2.5}$$

are rational functions of x, it is to be expected that there is a connection between the theory of continued fractions and the Padé table. In fact, if (2.5) is the continued fraction expansion, which may be derived by a number of methods one of which will explicitly be described in a later section, of the power series

$$\sum_{r=0}^{\infty} c_{m+r}x^r \tag{2.6}$$

then [2, p. 447] the quotients

$$\frac{U_{0,k}(x)}{V_{0,k}(x)},\quad \frac{U_{0,k+1}(x)}{V_{0,k+1}(x)},\quad \frac{U_{1,k+1}(x)}{V_{1,k+1}(x)},\quad \frac{U_{1,k+2}(x)}{V_{1,k+2}(x)},\quad \frac{U_{2,k+2}(x)}{V_{2,k+2}(x)},\ \cdots \tag{2.7}$$

are the successive convergents of the continued fraction

$$c_0 + c_1x + \cdots + c_kx^k + \frac{c_{k+1}x^{k+1}}{1-}\ \frac{q_1^{(k+1)}x}{1-}\ \frac{e_1^{(k+1)}x}{1-}\ \frac{q_2^{(k+1)}x}{1-}\ \frac{e_2^{(k+1)}x}{1-}\ \cdots, \tag{2.8}$$

while the quotients

$$\frac{U_{k,0}(x)}{V_{k,0}(x)},\quad \frac{U_{k,1}(x)}{V_{k,1}(x)},\quad \frac{U_{k+1,1}(x)}{V_{k+1,1}(x)},\quad \frac{U_{k+1,2}(x)}{V_{k+1,2}(x)},\quad \frac{U_{k+2,2}(x)}{V_{k+2,2}(x)},\ \cdots \tag{2.9}$$

are the successive convergents of the continued fraction

$$\frac{1}{d_0 + d_1x + \cdots + d_kx^k +}\ \frac{d_{k+1}x^{k+1}}{1-}\ \frac{q_1^{(k+1)'}x}{1-}\ \frac{e_1^{(k+1)'}x}{1-}\ \frac{q_2^{(k+1)'}x}{1-}\ \frac{e_2^{(k+1)'}x}{1-}\ \cdots \tag{2.10}$$

where, formally

$$\sum_{r=0}^{\infty} d_{k+r}x^r = \frac{d_k}{1-}\ \frac{q_1^{(k)'}x}{1-}\ \frac{e_1^{(k)'}x}{1-}\ \frac{q_2^{(k)'}x}{1-}\ \frac{e_2^{(k)'}x}{1-}\ \cdots \tag{2.11}$$

and

$$\left(\sum_{r=0}^{\infty} c_rx^r\right)\left(\sum_{r=0}^{\infty} d_rx^r\right) = 1. \tag{2.12}$$

Before proceeding further it will be stated that rational approximations will be derived which relate to the series

$$\beta(x) = \sum_{r=0}^{\infty} c_r x^r \tag{2.13}$$

and others which relate to the series

$$F(z) = \sum_{r=0}^{\infty} c_r z^{-r-1}. \tag{2.14}$$

Since these series may be transformed into each other by means of the simple relationships

$$\beta(x) = x^{-1}F(x^{-1}) \tag{2.15}$$

$$F(z) = z^{-1}\beta(z^{-1}), \tag{2.16}$$

it seems to be most economical to refer all the working to one of the series and further, it would seem to be a cause for surprise that this should be a matter for discussion. However, if the series (2.14) is rewritten as a power series in $x = z^{-1}$ it will be observed that the constant term is missing, i.e., that the condition that all the Hankel determinants (2.4) should be non-zero is violated. If on the other hand the series (2.13) is treated with methods to be described, there results such a wilful perversion of certain classical results, such as occur, for example, in the theory of orthogonal polynomials, as would encourage some other course of action to be adopted. A yet more decisive reason for difficulty is contained in the observation that the Padé table is essentially a symmetric structure in the sense that the power series expansion of $\{\beta(x)\}^{-1}$ has the same form as that of $\beta(x)$, whereas the power series expansion of $\{F(z)\}^{-1}$ has not the same form as that of $F(z)$.

Accordingly, the rational expressions $E_n^{(m)}(z)$ of the form

$$E_n^{(m)}(z) = \frac{r_n^{(m)}(z)}{p_n^{(m)}(z)} \tag{2.17}$$

where $p_n^{(m)}(z)$ is the polynomial of the nth degree

$$p_n^{(m)}(z) = \sum_{s=0}^{n} k_{n,s}^{(m)} z^s, \tag{2.18}$$

$r_n^{(m)}(z)$ is a function of the form

$$r_n^{(m)}(z) = z^{-m} \sum_{s=0}^{m+n-1} t_{n,s}^{(m)} z^s, \qquad r_0^{(0)}(z) = 0, \tag{2.19}$$

and the series expansion of $E_n^{(m)}(z)$ in inverse powers of z agree with that of $F(z)$ as far as the term containing z^{-m-2n}, will be arranged in the following array:

$$\begin{array}{llllll}
E_0^{(0)}(z) & & & & & \\
E_0^{(1)}(z) & E_1^{(0)}(z) & & & & \\
E_0^{(2)}(z) & E_1^{(1)}(z) & E_2^{(0)}(z) & & & \\
E_0^{(3)}(z) & E_1^{(2)}(z) & E_2^{(1)}(z) & E_3^{(0)}(z) & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
E_0^{(m)}(z) & E_1^{(m-1)}(z) & E_2^{(m-2)}(z) & E_3^{(m-3)}(z) & \cdots & E_m^{(0)}(z)
\end{array}$$

in which the quantity $E_n^{(m)}(z)$ occurs at the intersection of the $(n+1)$th column and the $(m+1)$th diagonal. This array will be referred to throughout this paper as the E-array. Clearly the constituents $p_n^{(m)}(z)$, $r_n^{(m)}(z)$ of the function $E_n^{(m)}(z)$ may also be displayed in the same array. If, in the notation of equation (2.18), the restriction

$$k_{n,n}^{(m)} = 1 \tag{2.20}$$

is imposed, it is clear that

$$z^{-n}p_n^{(m)}(z) = V_{n,m+n-1}(x),\ zx = 1 \tag{2.21}$$

and that

$$z^{-n+1}r_n^{(m)}(z) = U_{n,m+n-1}(x),\ zx = 1. \tag{2.22}$$

If, as is convenient from the point of view of numerical computation, the Padé table is regarded as an array of vector pairs

$$(\alpha_{\mu,0}, \alpha_{\mu,1}, \cdots, \alpha_{\mu,\nu}), \qquad (\beta_{0,\nu}, \beta_{1,\nu}, \cdots, \beta_{\mu,\nu}), \tag{2.23}$$

so that

$$U_{\mu,\nu}(x) = (\alpha_{\mu,0}, \alpha_{\mu,1}, \cdots, \alpha_{\mu,\nu})(1, x, \cdots, x^{\nu})^T \tag{2.24}$$

and

$$V_{\mu,\nu}(x) = (\beta_{0,\nu}, \beta_{1,\nu}, \cdots, \beta_{\mu,\nu})(1, x, \cdots, x^{\mu})^T, \tag{2.25}$$

and the E-array is regarded as an array of vector pairs

$$(k_{n,0}^{(m)}, k_{n,1}^{(m)}, \cdots, k_{n,n}^{(m)}), \qquad (i_{n,0}^{(m)}, i_{n,1}^{(m)}, \cdots, i_{m+n-1}^{(m)})$$

so that

$$p_n^{(m)}(z) = (k_{n,0}^{(m)}, k_{n,1}^{(m)}, \cdots, k_{n,n}^{(m)})(1, z, \cdots, z^n)^T$$

and

$$r_n^{(m)}(z) = z^{-m}(i_{n,0}^{(m)}, i_{n,1}^{(m)}, \cdots, i_{n,m+n-1}^{(m)})(1, z, \cdots, z^{m+n+1})^T$$

it is seen that the whole E-array may be obtained from the Padé table by transposing the Padé table, deleting these terms lying above the diagonal beginning with the second term of the first row, and placing the quantity $E_0^{(0)}(z) = 0$ at the peak of the array; conversely, part of the Padé table may be obtained from the E-array by removing the entry $E_0^{(0)}(z)$ and transposing the E-array about the diagonal $E_n^{(1)}(z)$, $n = 0, 1, \cdots$.

Approximations relating to both series (2.13) and (2.14) will be derived, the series chosen for treatment in a particular case being selected according to the dictates of convenience. The variable used will indicate the series under consideration. Methods for deriving rational approximation formulas will now be discussed.

3. The E-Array.

3.1 *Orthogonal Polynomials*

The various questions associated with the central problem of obtaining rational functions from power series expansions are unified and greatly clarified by an appeal

to the classical theory of orthogonal polynomials. This is usually introduced [4, p. 153] by focusing attention upon the integral formula

$$\int_a^b p_r(t)p_s(t)\,d\phi(t) = \begin{cases} 0 & r \neq s \\ \omega_s & r = s \end{cases} \tag{3.1.1}$$

which serves uniquely to define a sequence of polynomials $p_r(t)$ $r = 0, 1, \cdots$, and it is then noted at some later stage that if

$$\int_a^b t^s\,d\phi(t) = c_s \tag{3.1.2}$$

then formally

$$\int_a^b \frac{1}{z-t}\,d\phi(t) = \sum_{r=0}^{\infty} c_r z^{-r-1}. \tag{3.1.3}$$

Since, however, the moments c_s are the central feature of the problem in hand, and the introduction of the integral and weight function $\phi(t)$ (in view of the difficulty of envisaging an appropriate form for $\phi(t)$) may even serve as an impediment to understanding, the present treatment will slightly be varied.

Accordingly, define a process I by

$$I(t^s) = c_s \qquad s = 0, 1, \cdots \tag{3.1.4}$$

and construct a sequence of orthogonal polynomials $p_n^{(m)}(z)$ $n = 0, 1, \cdots, m = 0, 1, \cdots$, from the relations

$$I(t^{m+s}p_n^{(m)}(t)) = \begin{cases} 0 & 0 \leqq s \leqq n-1 \\ \omega_n & s = n \end{cases} \tag{3.1.5}$$

where ω_n is so chosen that in the representation

$$p_n^{(m)}(z) = \sum_{s=0}^{n} k_{n,s}^{(m)} z^s \tag{3.1.6}$$

$$k_{n,n}^{(m)} = 1 \tag{3.1.7}$$

Construct a second sequence of polynomials

$$o_n^{(m)}(z) = \sum_{s=0}^{n-1} j_{n,s}^{(m)} z^s \tag{3.1.8}$$

from the relation

$$o_n^{(m)}(z) = I\left\{\frac{t^m(p_n^{(m)}(z) - p_n^{(m)}(t))}{z-t}\right\} \qquad n = 1, 2, \cdots; m = 0, 1, \cdots \tag{3.1.9}$$

These are the classical associated polynomials occurring in the theory of orthogonal polynomials [4, p. 162].

Note in passing that formally

$$I\left\{\frac{t^m}{z-t}\right\} = \sum_{s=0}^{\infty} c_{m+s} z^{-s-1}. \tag{3.1.10}$$

Consider the quotient $o_n^{(m)}(z)/p_n^{(m)}(z)$, there follows

$$\frac{o_n^{(m)}(z)}{p_n^{(m)}(z)} = I\left\{\frac{t^m(p_n^{(m)}(z) - p_n^{(m)}(t))}{z-t}\right\} \Big/ p_n^{(m)}(z)$$

$$= I\left\{\frac{t^m}{z-t}\right\} - \frac{1}{p_n^{(m)}(z)} I\left\{\frac{t^m p_n^{(m)}(t)}{z-t}\right\}$$

$$= \sum_{s=0}^{\infty} c_{m+s} z^{-s-1} - \{z^{-n} - k_{n,n-1}^{(m)} z^{-n-1} + \cdots\} I\left\{\sum_{s=0}^{\infty} p_n^{(m)}(t) t^{m+s} z^{-s-1}\right\} \tag{3.1.11}$$

$$= \sum_{s=0}^{\infty} c_{m+s} z^{-s-1} + P_{2n+1}(z)$$

where $P_{2n+1}(z)$ is a power series beginning with a term in z^{-2n-1}. Define a third sequence of functions $r_n^{(m)}(z)$, where

$$z^m r_n^{(m)}(z) = \sum_{s=0}^{m+n-1} t_{n,s}^{(m)} z^s \tag{3.1.12}$$

by means of the relation

$$r_n^{(m)}(z) = I\left\{\frac{z^m p_n^{(m)}(z) - t^m p_n^{(m)}(t)}{z-t}\right\}, \tag{3.1.13}$$

and note that

$$\frac{r_n^{(m)}(z)}{p_n^{(m)}(z)} = \sum_{s=0}^{m-1} c_s z^{-s-1} + z^{-m} \frac{o_n^{(m)}(z)}{p_n^{(m)}(z)} = \sum_{s=0}^{\infty} c_s z^{-s-1} + P_{m+2n+1}(z) \tag{3.1.14}$$

Thus it can be seen that the construction of the E-array may be greatly facilitated if the functions $o_n^{(m)}(z)$, $p_n^{(m)}(z)$ and $r_n^{(m)}(z)$ may be constructed with moderate ease. Results which enable this to be done, and others which will be of later interest, are now given.

Eliminating the coefficients $k_{n,s}^{(m)}$ $s = 0, 1, \cdots, n$, between n equations of the form (3.1.5) with $s = 0, 1, \cdots, n-1$, (3.1.6) amd (3.1.7) there follows the determinantal formula [4, p. 158] for the polynomial $p_n^{(m)}(z)$

$$p_n^{(m)}(z) = (-1)^n \begin{vmatrix} 1 & z & \cdots & z^n \\ c_m & c_{m+1} & \cdots & c_{m+n} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+n+1} \\ \vdots & \vdots & & \vdots \\ c_{m+n-1} & c_{m+n} & \cdots & c_{m+2n-1} \end{vmatrix} \Big/ \begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+n-1} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+n} \\ \vdots & \vdots & & \vdots \\ c_{m+n-1} & c_{m+n} & \cdots & c_{m+2n-2} \end{vmatrix} \tag{3.1.15}$$

Further it may be shown [4, p. 158] that the polynomials satisfy a recursion of the form

$$p_{r+1}^{(m)}(z) = (z - \alpha_r^{(m)}) p_r^{(m)}(z) - \beta_{r-1}^{(m)} p_{r-1}^{(m)}(z)$$
$$p_0^{(m)}(z) = 1, \qquad p_1^{(m)}(z) = z - \alpha_0^{(m)}, \tag{3.1.16}$$

and, from the definitions (3.1.9) and (3.1.13)

$$o_{r+1}^{(m)}(z) = (z - \alpha_r^{(m)})o_r^{(m)}(z) - \beta_{r-1}^{(m)}o_{r-1}^{(m)}(z)$$
$$o_0^{(m)}(z) = 0, \qquad o_1^{(m)}(z) = c_m \tag{3.1.17}$$

and

$$r_{r+1}^{(m)}(z) = (z - \alpha_r^{(m)})r_r^{(m)}(z) - \beta_{r-1}^{(m)}r_{r-1}^{(m)}(z)$$
$$r_0^{(m)}(z) = \sum_{s=0}^{m-1} c_s z^{-s-1}, \qquad r_1^{(m)}(z) = p_1^{(m)}(z)r_0^{(m)}(z) + c_m z^{-m}. \tag{3.1.18}$$

These recursions, in conjunction with equations (3.1.11) and (3.1.14) serve to identify the quotient $o_n^{(m)}(z)/p_n^{(m)}(z)$ as the nth convergent of the continued fraction

$$F_m(z) = \frac{c_m}{z - \alpha_0^{(m)} -} \frac{\beta_0^{(m)}}{z - \alpha_1^{(m)} -} \cdots \frac{\beta_{r-1}^{(m)}}{z - \alpha_r^{(m)} -} \cdots \tag{3.1.19}$$

where formally

$$F_m(z) = \sum_{r=0}^{\infty} c_{m+r} z^{-r-1}, \tag{3.1.20}$$

and the quotient $r_n^{(m)}(z)/p_n^{(m)}(z)$ as the nth convergent of the continued fraction

$$F(z) = \sum_{s=0}^{m-1} c_s z^{-s-1} + z^{-m}\left\{\frac{c_m}{z - \alpha_0^{(m)} -} \frac{\beta_0^{(m)}}{z - \alpha_1^{(m)} -} \cdots - \frac{\beta_{r-1}^{(m)}}{z - \alpha_r^{(m)} -} \cdots\right\} \tag{3.1.21}$$

Multiplying equation (3.1.16) through by t^{m+r-1} and replacing z by t, it follows, by virtue of equation (3.12), that

$$I(t^{m+r}p_r^{(m)}(t)) = \beta_{r-1}^{(m)}I(t^{m+r-1}p_{r-1}^{(m)}(t)) = c_m\beta_0^{(m)}\beta_1^{(m)} \cdots \beta_{r-1}^{(m)} \tag{3.1.22}$$

Similarly, by multiplying the same equation through by t^{m+r}, it follows that

$$I(t^{m+r+1}p_r^{(m)}(t)) = c_m\beta_0^{(m)}\beta_1^{(m)} \cdots \beta_{r-1}^{(m)}(\alpha_0^{(m)} + \alpha_1^{(m)} + \cdots + \alpha_r^{(m)}) \tag{3.1.23}$$

Equations (3.1.16), (3.1.22), and (3.1.23), taken in conjunction are now sufficient to determine in succession the quantities $k_{r,s}^{(m)}$, $\beta_{r-1}^{(m)}$, $\alpha_r^{(m)}$ for any preassigned value of m [3, p. 196]. The coefficients $t_{r,s}^{(m)}$ may be recovered by means of the recursion (3.1.18), or from the condition (3.1.14), or from the recursion (3.1.17) in conjunction with the condition

$$r_n^{(m)}(z) = p_n^{(m)}(z) \sum_{s=0}^{m-1} c_s z^{-s-1} + z^{-m}o_n^{(m)}(z). \tag{3.1.24}$$

3.2 *Stieltjes' Algorithm* [5]; [2, p. 326]; [3, p. 203].

An algorithm for obtaining the quantities $\alpha_r^{(m)}$, $\beta_{r-1}^{(m)}$ and hence, by virtue of equations (3.1.16) and (3.1.18), the quotients $r_n^{(m)}(z)/p_n^{(m)}(z)$, is implicit in the followng theorem due to Stieltjes: If the bilinear form $\sum_{p,q=0}^{\infty} c_{m+p+q}x_p y_q$ is decomposed into

$$\sum_{p,q=0}^{\infty} c_{m+p+q}x_p y_q = c_m(h_{0,0}^{(m)}x_0 + h_{1,0}^{(m)}x_1 + \cdots)(h_{0,0}^{(m)}y_0 + h_{1,0}^{(m)}y_1 + \cdots)$$
$$+ c_m\beta_0^{(m)}(h_{1,1}^{(m)}x_1 + h_{2,1}^{(m)}x_2 + \cdots)(h_{1,1}^{(m)}y_1 + h_{2,1}^{(m)}y_2 + \cdots)$$
$$+ c_m\beta_0^{(m)}\beta_1^{(m)}(h_{2,2}^{(m)}x_2 + h_{3,2}^{(m)}x_3 + \cdots)(h_{2,2}^{(m)}y_2 + h_{3,2}^{(m)}y_3 + \cdots) + \cdots$$

then the quantities $h_{r,s}^{(m)}$ satisfy recursions involving quantities $\alpha_r^{(m)}$, $\beta_{r-1}^{(m)}$ expressible by the matrix equation

$$\begin{pmatrix} h_{0,0}^{(m)} & 0 & 0 & \cdots \\ h_{1,0}^{(m)} & h_{1,1}^{(m)} & 0 & \cdots \\ h_{2,0}^{(m)} & h_{2,1}^{(m)} & h_{2,2}^{(m)} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} -\alpha_0^{(m)} & 1 & 0 & 0 & \cdots \\ \beta_0^{(m)} & -\alpha_1^{(m)} & 1 & 0 & \cdots \\ 0 & \beta_1^{(m)} & -\alpha_2^{(m)} & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{3.2.2}$$

$$= \begin{pmatrix} h_{1,0}^{(m)} & h_{1,1}^{(m)} & 0 & \cdots \\ h_{2,0}^{(m)} & h_{2,1}^{(m)} & h_{2,2}^{(m)} & \cdots \\ h_{3,0}^{(m)} & h_{3,1}^{(m)} & h_{3,2}^{(m)} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

in which

$$h_{r,r}^{(m)} = 1 \qquad r = 0, 1, \cdots, \tag{3.2.3}$$

furthermore, by equating the coefficients of $x_p y_q$ on the right and left hand sides of equation (3.2.1)

$$c_{m+p+q} = c_m h_{p,0}^{(m)} h_{q,0}^{(m)} + c_m \beta_0^{(m)} h_{p,1}^{(m)} h_{q,1}^{(m)} + c_m \beta_0^{(m)} \beta_1^{(m)} h_{p,2}^{(m)} h_{q,2}^{(m)} + \cdots \tag{3.2.4}$$

the series in (3.2.4) being terminated by observing that $h_{r,s}^{(m)} = 0$ if $s > r$. Finally, from equation (3.2.2)

$$\alpha_0^{(m)} = -h_{1,0}^{(m)} \qquad \alpha_r^{(m)} = h_{r,r-1}^{(m)} - h_{r+1,r}^{(m)} \qquad r = 1, 2, \cdots \tag{3.2.5}$$

The significance of this result is revealed by denoting the triangular matrix in equation (3.2.2) by $\mathbf{H}^{(m)}$, the matrix

$$\begin{pmatrix} c_m & c_{m+1} & c_{m+2} & \cdots \\ c_{m+1} & c_{m+2} & c_{m+3} & \cdots \\ c_{m+2} & c_{m+3} & c_{m+4} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{3.2.6}$$

by $\mathbf{C}^{(m)}$, the diagonal matrix

$$\begin{pmatrix} c_m & 0 & 0 & \cdots \\ 0 & c_m \beta_0^{(m)} & 0 & \cdots \\ 0 & 0 & c_m \beta_0^{(m)} \beta_1^{(m)} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{3.2.7}$$

by $\mathbf{D}^{(m)}$, the vector $(x_0, x_1, x_2, \cdots)$ by $\mathbf{x}$, the vector $(y_0, y_1, y_2, \cdots)^T$ by $\mathbf{y}$, and rewriting equation (3.2.1) as

$$\mathbf{x}\mathbf{C}^{(m)}\mathbf{y} = \mathbf{x}\mathbf{H}^{(m)}\mathbf{D}^{(m)}\mathbf{H}^{(m)T}\mathbf{y}. \tag{3.2.8}$$

Thus it is seen immediately that the process of determining the triangular array $\mathbf{H}^{(m)}$ is equivalent to the familiar Gauss-Banachiewicz decomposition for solving linear systems, in which the matrix $\mathbf{D}^{(m)}$ is so chosen that the diagonal elements of the matrix $\mathbf{H}^{(m)}$ are unity (see the remark contained upon page 278 of [6]). The relationship between the matrix $\mathbf{H}^{(m)}$ and the matrix

$$\mathbf{K}^{(m)} = \begin{pmatrix} k_{0,0}^{(m)} & 0 & 0 & \cdots \\ k_{1,0}^{(m)} & k_{1,1}^{(m)} & 0 & \cdots \\ k_{2,0}^{(m)} & k_{2,1}^{(m)} & k_{2,2}^{(m)} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{3.2.9}$$

of coefficients in the orthogonal polynomials $p_n^{(m)}(z)$ $n = 0, 1, \cdots$ is revealed by noting that a consequence of equations (3.1.7) and (3.1.22) taken in conjunction is that

$$I(t^m p_r^{(m)}(t) p_s^{(m)}(t)) = \begin{cases} 0 & r = 0, 1, \cdots, s-1, \\ c_m \beta_0^{(m)} \beta_1^{(m)} \cdots \beta_{r-1}^{(m)} & r = s \end{cases} \tag{3.2.10}$$

or in matrix notation

$$\mathbf{K}^{(m)} \mathbf{C}^{(m)} \mathbf{K}^{(m)T} = \mathbf{D}^{(m)} \tag{3.2.11}$$

Using the result (3.2.8) it follows that

$$\mathbf{K}^{(m)} \mathbf{H}^{(m)} \mathbf{D}^{(m)} \mathbf{H}^{(m)T} \mathbf{K}^{(m)T} = \mathbf{D}^{(m)} \tag{3.2.12}$$

or

$$\mathbf{K}^{(m)} = \mathbf{H}^{(m)-1}. \tag{3.2.13}$$

The author is indebted to Professors F. L. Bauer and H. Rutishauser for remarks upon this section.

3.3 *The Quotient Difference Algorithm*

If the substitutions

$$a_s^{(m)} = q_{s+1}^{(m)} + e_s^{(m)} \qquad \beta_s^{(m)} = e_{s+1}^{(m)} q_{s+1}^{(m)} \qquad s = 0, 1, \cdots$$

with

$$e_0^{(m)} = 0$$

are made, the continued fraction (3.1.19) becomes

$$\frac{c_m}{z - q_1^{(m)} -} \frac{e_1^{(m)} q_1^{(m)}}{z - q_2^{(m)} - e_1^{(m)} -} \cdots \frac{e_r^{(m)} q_r^{(m)}}{z - q_{r+1}^{(m)} - e_r^{(m)} -} \cdots \tag{3.3.2}$$

and may be recognized as the even part [2, p. 197], [3, p. 20],

$$\frac{c_m}{z -} \frac{q_1^{(m)}}{1 -} \frac{e_1^{(m)}}{z -} \frac{q_2^{(m)}}{1 -} \frac{e_2^{(m)}}{z -} \cdots. \tag{3.3.3}$$

The odd part of (3.3.3) is

$$\frac{c_m}{z}\left\{1 + \frac{q_1^{(m)}}{z - q_1^{(m)} - e_1^{(m)} -} \frac{e_1^{(m)} q_2^{(m)}}{z - q_2^{(m)} - e_2^{(m)} -} \frac{e_r^{(m)} q_{r+1}^{(m)}}{z - q_{r+1}^{(m)} - e_{r+1}^{(m)} -} \cdots\right\}. \tag{3.3.4}$$

However

$$\sum_{r=0}^{\infty} c_{m+r} z^{-r-1} = \frac{c_m}{z} + z^{-1} \sum_{r=0}^{\infty} c_{m+r+1} z^{-r-1} \tag{3.3.5}$$

and hence

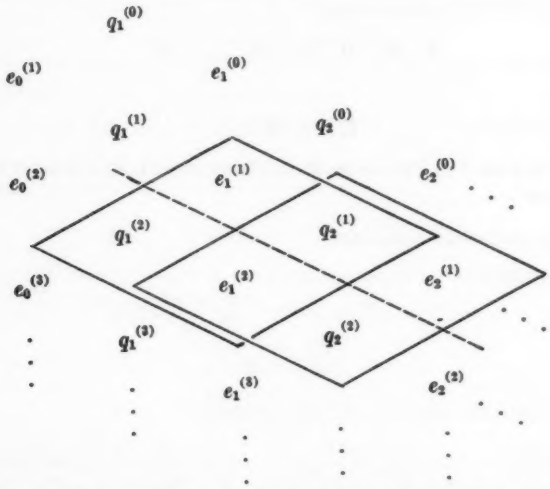
$$\sum_{r=0}^{\infty} c_{m+r+1} z^{-r-1} = \frac{c_m q_1^{(m)}}{z - q_1^{(m)} - e_1^{(m)} -} \frac{e_1^{(m)} q_2^{(m)}}{z - q_2^{(m)} - e_2^{(m)} -} \cdots \frac{e_r^{(m)} q_{r+1}^{(m)}}{z - q_{r+1}^{(m)} - e_{r+1}^{(m)} -} \cdots . \tag{3.3.6}$$

Substituting $m + 1$ for m in (3.3.2) and equating the coefficients in the resulting continued fraction with those of (3.3.6) there follows the two systems of equations

$$q_r^{(m)} + e_r^{(m)} = q_r^{(m+1)} + e_{r-1}^{(m+1)} \tag{3.3.7}$$

$$q_{r+1}^{(m)} e_r^{(m)} = q_r^{(m+1)} e_r^{(m+1)} \tag{3.3.8}$$

which may progressively be used to derive the coefficients in the continued fraction (3.3.2) from those of the infinite series $\sum_{r=0}^{\infty} c_{m+r} z^{-r-1}$
The quantities $q_r^{(m)}$, $e_r^{(m)}$ may be arranged in the following scheme:



the various quantities occurring in equations (3.3.7) and (3.3.8) are then seen to be located at the corners of a system of lozenges, two of which have been sketched. The computations are started with the initial conditions

$$e_0^{(m)} = 0, \qquad q_1^{(m)} = c_{m+1}/c_m . \tag{3.3.9}$$

The system of equations (3.3.7) and (3.3.8) is the Quotient-Difference (q-d) algorithm. Formulas relating to it have been given by a number of authors [7], [8], [3, p. 382], [9], [10], [11]. Properties of this algorithm which are of critical interest in the theory of numerical analysis have been exploited by Rutishauser in his expository study [10]. An interesting generalization of the algorithm has recently been given by Bauer [12]. Once the quantities $q_s^{(m)}$, $e_s^{(m)} s = 1, 2, \cdots$ have been derived, the system of polynomials $o_s^{(m)}(z)$, $p_s^{(m)}(z)$ and the functions for a fixed value of m, may be constructed by use of equations (3.3.1), (3.1.16), (3.1.17), and (3.1.18).

3.4 *Further Polynomial Recursion Systems*

Two further results which relate the quantities $q_r^{(m)}$, $e_r^{(m)}$ and the polynomials $p_r^{(m)}(z)$ are [10, p. 11]

$$p_r^{(m+1)}(z) = p_r^{(m)}(z) - e_r^{(m)} p_{r-1}^{(m+1)}(z) \tag{3.4.1}$$

and

$$p_r^{(m)}(z) = z p_{r-1}^{(m+1)}(z) - q_r^{(m)} p_{r-1}^{(m)}(z) \tag{3.4.2}$$

It follows trivially from the definitions (3.1.9) and (3.1.13) that

$$z o_r^{(m)}(z) - o_r^{(m+1)}(z) - e_r^{(m)} o_{r-1}^{(m+1)}(z) = c_m p_r^{(m)}(z) \tag{3.4.3}$$

$$o_r^{(m)}(z) + q_r^{(m)} o_{r-1}^{(m)}(z) - o_{r-1}^{(m+1)}(z) = c_m p_{r-1}^{(m+1)}(z) \tag{3.4.4}$$

$$r_r^{(m+1)}(z) = r_r^{(m)}(z) - e_r^{(m)} r_{r-1}^{(m+1)}(z) \tag{3.4.5}$$

$$r_r^{(m)}(z) = z r_{r-1}^{(m+1)}(z) - q_r^{(m)} r_{r-1}^{(m)}(z) \tag{3.4.6}$$

and after some manipulation, that

$$e_r^{(m)} z p_{r-1}^{(m+1)}(z) + (q_{r+1}^{(m)} - e_{r+1}^{(m-1)} - z) p_r^{(m)}(z) + p_{r+1}^{(m-1)}(z) = 0 \tag{3.4.7}$$

$$q_{r+1}^{(m-1)} p_r^{(m-1)}(z) + (e_r^{(m)} - q_{r+1}^{(m-1)} - z) p_r^{(m)}(z) + z p_r^{(m+1)}(z) = 0 \tag{3.4.8}$$

$$e_r^{(m)} z r_{r-1}^{(m+1)}(z) + (q_{r+1}^{(m)} - e_{r+1}^{(m+1)} - z) r_r^{(m)}(z) + r_{r+1}^{(m-1)}(z) = 0 \tag{3.4.9}$$

$$q_{r+1}^{(m-1)} r_r^{(m-1)}(z) + (e_r^{(m)} - q_{r+1}^{(m-1)} - z) r_r^{(m)}(z) + z r_r^{(m+1)}(z) = 0 \tag{3.4.10}$$

$$\begin{aligned} e_r^{(m)} o_{r-1}^{(m+1)}(z) + (q_{r+1}^{(m)} - e_{r+1}^{(m-1)} - z) o_r^{(m)}(z) + z o_{r+1}^{(m-1)}(z) \\ + c_m e_r^{(m)} p_{r-1}^{(m+1)}(z) - c_{m-1} p_{r+1}^{(m-1)}(z) = 0 \end{aligned} \tag{3.4.11}$$

$$\begin{aligned} q_r^{(m-1)} z o_r^{(m-1)}(z) + (e_r^{(m)} - q_r^{(m-1)} z) o_r^{(m)}(z) + o_r^{(m+1)}(z) \\ + c_m p_r^{(m)}(z) - c_{m-1} q_r^{(m-1)} p_r^{(m-1)}(z) = 0. \end{aligned} \tag{3.4.12}$$

The transcribed recursions (3.1.16) and (3.1.18), namely

$$p_{r+1}^{(m)}(z) = \{z - q_{r+1}^{(m)} - e_r^{(m)}\} p_r^{(m)}(z) - e_r^{(m)} q_r^{(m)} p_{r-1}^{(m)}(z) \tag{3.4.13}$$

$$r_{r+1}^{(m)}(z) = \{z - q_{r+1}^{(m)} - e_r^{(m)}\} r_r^{(m)}(z) - e_r^{(m)} q_r^{(m)} r_{r-1}^{(m)}(z) \tag{3.4.14}$$

are recursions involving quantities lying upon diagonals of the E-array, (3.4.7) and (3.4.9) and (3.4.10) involve quantities lying upon rows and columns of the E-array respectively. Both pairs of recursions (3.4.7) and (3.4.9), and (3.4.8) and (3.4.10) could be used to construct continued fraction expansions whose convergents lie upon a row or a column of the E-array, and indeed, by suitable use of the recursions (3.4.1), (3.4.2), (3.4.5) and (3.4.6), continued fractions may be constructed whose convergents lie upon arbitrary paths in the E-array, but numerical evidence shows that practical interest attaches only to the continued fractions of the form (3.3.2).

3.5 *The Euler-Minding Relations*

Eliminating the quantity $z - \alpha_r^{(m)}$ between the recursions

$$p_{r+1}^{(m)}(z) = (z - \alpha_r^{(m)})p_r^{(m)}(z) - \beta_{r-1}^{(m)}p_{r-1}^{(m)}(z) \tag{3.5.1}$$

$$r_{r+1}^{(m)}(z) = (z - \alpha_r^{(m)})r_r^{(m)}(z) - \beta_{r-1}^{(m)}r_{r-1}^{(m)}(z) \tag{3.5.2}$$

there follows

$$\begin{aligned} r_{r+1}^{(m)}(z)p_r^{(m)}(z) - p_{r+1}^{(m)}(z)r_r^{(m)}(z) &= \beta_{r-1}^{(m)}\{r_r^{(m)}(z)p_{r-1}^{(m)}(z) \\ &\quad - r_{r-1}^{(m)}(z)p_r^{(m)}(z)\} \\ &= c_m\beta_0^{(m)}\beta_1^{(m)} \cdots \beta_{r-1}^{(m)} \end{aligned} \tag{3.5.3}$$

or, dividing equation (3.5.3) through by $p_{r+1}^{(m)}(z)p_r^{(m)}(z)$ writing, $1, 2 \cdots n$ for $r + 1$, and adding the resulting equations

$$\frac{r_n^{(m)}(z)}{p_n^{(m)}(z)} = r_0^{(m)}(z) + \sum_{r=1}^{r=n} \frac{c_m \beta_0^{(m)} \cdots \beta_{r-2}^{(m)}}{p_{r-1}^{(m)}(z)p_r^{(m)}(z)} \tag{3.5.4}$$

the Euler-Minding relations [2, p. 16], [3, p. 17], for the continued fraction (3.1.21).

3.6 *A Non-linear Recursion*

The rational function $E_n^{(m)}(z)$ may also be computed by using a further result in the theory of continued fractions, namely that nth convergent

$$C_n = \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots \frac{a_n}{b_n} \tag{3.6.1}$$

of the continued fraction

$$\frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots \frac{a_n}{b_n +} \cdots \tag{3.6.1}$$

may be computed by evaluating the sequence

$$D_1 = \frac{a_n}{b_n}, \qquad D_{r+1} = \frac{a_{n-r}}{b_{n-r} + D_r} \qquad r = 1, 2, \cdots, n-1, \tag{3.6.3}$$

when

$$D_n = C_n .$$

Writing

$$D_{\nu,s}^{(m)}(z) = \frac{d_{\nu,s,0}^{(m)} + d_{\nu,s,1}^{(m)} z + \cdots + d_{\nu,s,s-1}^{(m)} z^{s-1}}{d_{\nu,s,0}^{(m)'} + d_{\nu,s,1}^{(m)'} z + \cdots + d_{\nu,s,s}^{(m)'} z^s} \tag{3.6.4}$$

it is seen that the rational function $o_\nu^{(m)}(z)/p_\nu^{(m)}(z)$ may be constructed by evaluating the sequence

$$\begin{aligned} D_{\nu,1}^{(m)}(z) &= \frac{\beta_{\nu-2}^{(m)}}{z - \alpha_{\nu-1}^{(m)}} \\ D_{\nu,s+1}^{(m)}(z) &= \frac{\beta_{\nu-s-2}^{(m)}}{z - \alpha_{\nu-s-1}^{(m)} - D_{\nu,s}^{(m)}(z)} \end{aligned} \tag{3.6.5}$$

when

$$D_{\nu,\nu}^{(m)}(z) = o_\nu^{(m)}(z)/p_\nu^{(m)}(z).$$

The quotient $E_n^{(m)}(z)$ may then be constructed by means of equation (3.1.24).

It will be observed however, that whereas the evaluation of the sequences (3.1.16), (3.1.18) and (3.5.4) are progressive in the sense that at each stage there results the evaluation of a new function $p_r^{(m)}(z)$, $r_r^{(m)}(z)$ or $E_r^{(m)}(z)$, the evaluation of the whole sequence $D_{r,s}^{(m)}(z)$ $s = 1, \cdots, \nu$ results in the computation of but one quotient $E_\nu^{(m)}(z)$.

3.7 *The $e_m(S_n)$ Transformation* [13] [14]

If the partial sums S_m of the infinite series $\sum_{r=0}^{\infty} c_r z^{-r-1}$ are defined by

$$S_m = \sum_{r=0}^{m-1} c_r z^{-r-1} \qquad r = 1, 2, \cdots, S_0 = 0 \tag{3.7.1}$$

then

$$\begin{aligned} \Delta S_m &= S_{m+1} - S_m \\ &= c_m z^{-m-1}. \end{aligned} \tag{3.7.2}$$

Equation (2.17), by means of suitable operations upon rows and columns of the determinantal expressions which arise from the application of equation (3.1.13) to the expression (3.1.15), may be manipulated into the form

$$E_n^{(m)}(z) = \frac{\begin{vmatrix} S_m & S_{m+1} & \cdots & S_{m+n} \\ \Delta S_m & \Delta S_{m+1} & \cdots & \Delta S_{m+n} \\ \Delta S_{m+1} & \Delta S_{m+2} & \cdots & \Delta S_{m+n+1} \\ \vdots & \vdots & \cdots & \vdots \\ \Delta S_{m+n-1} & \Delta S_{m+n} & \cdots & \Delta S_{m+2n-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_m & \Delta S_{m+1} & \cdots & \Delta S_{m+n} \\ \Delta S_{m+1} & \Delta S_{m+2} & \cdots & \Delta S_{m+n+1} \\ \vdots & \vdots & \cdots & \vdots \\ \Delta S_{m+n-1} & \Delta S_{m+n} & \cdots & \Delta S_{m+2n-1} \end{vmatrix}} \tag{3.7.3}$$

The determinantal quotient (3.7.3) was denoted by Shanks [14] as $e_n(S_{m+n})$ and was investigated, primarily as a non-linear sequence to sequence transformation, by him.

3.8 *The ϵ-Algorithm*

The various quantities $E_n^{(m)}(z)$ $m = 0, 1, \cdots$; $n = 0, 1, \cdots$, may be constructed by means of a simple recursive algorithm due to Wynn [15]. According to this, quantities $E_{n+1/2}^{(m)}(z)$ $n = -1, 0, 1, \cdots$ are introduced, and using as boundary values $E_{-1/2}^{(m)}(z) = 0$, $E_0^{(m)}(z) = S_m$, $m = 0, 1, \cdots$ further quantities $E_s^{(m)}(z)$ $s = \frac{1}{2}, 1, \frac{3}{2}, \cdots$; $m = 0, 1, \cdots$ are constructed by means of the recursion

$$E_s^{(m)}(z) = E_{s-1}^{(m+1)}(z) + \frac{1}{E_{s-1/2}^{(m+1)}(z) - E_{s-1/2}^{(m)}(z)} \tag{3.8.1}$$

(In the original exposition of this algorithm, [15], the right side of (3.7.3) was regarded as the result of a non-linear transformation of the quantities S_r, and the notation $E_{n+1/2}^{(m)}(z) = \epsilon_{2n+1}(S_m)$, $E_n^{(m)}(z) = \epsilon_{2n}(S_m)$, was used.)

Since all the quantities $E_s^{(m)}(z)$ are rational functions of z, and equation (3.8.1) may be manipulated into such a form as would require only the multiplication,

addition, and subtraction of polynomial expressions, the $\in$-algorithm provides a method for computing the constituents of the E-array.

A certain measure of economy in the use of the $\in$-algorithm for this purpose may however be effected by writing down explicit formulas for some of the quantities occurring in equation (3.8.1). Accordingly, as may easily be verified

$$E_{n+1}^{(m)}(z) - E_n^{(m+1)}(z) = \frac{1}{E_{n+1/2}^{(m+1)}(z) - E_{n+1/2}^{(m)}(z)}$$

$$= \frac{\begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+n} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+n+1} \\ \vdots & \vdots & & \vdots \\ c_{m+n} & c_{m+n+1} & \cdots & c_{m+2n} \end{vmatrix} \begin{vmatrix} c_{m+1} & c_{m+2} & \cdots & c_{m+n+1} \\ c_{m+2} & c_{m+3} & \cdots & c_{m+n+2} \\ \vdots & \vdots & & \vdots \\ c_{m+n+1} & c_{m+n+2} & \cdots & c_{m+2n+1} \end{vmatrix}}{\begin{vmatrix} 1 & z & \cdots & z^{n+1} \\ c_m & c_{m+1} & \cdots & c_{m+n+1} \\ \vdots & \vdots & & \vdots \\ c_{m+n} & c_{m+n+1} & \cdots & c_{m+2n+1} \end{vmatrix} \begin{vmatrix} 1 & z & \cdots & z^n \\ c_{m+1} & c_{m+2} & \cdots & c_{m+n+1} \\ \vdots & \vdots & & \vdots \\ c_{m+n} & c_{m+n+1} & \cdots & c_{m+2n} \end{vmatrix}} \tag{3.8.2}$$

$$= \frac{c_{m+1} \prod_{\mu=1}^{n} e_\mu^{(m+1)} q_\mu^{(m+1)}}{p_{n+1}^{(m)}(z) p_n^{(m+1)}(z)}$$

$$= \frac{c_{m+1} \beta_0^{(m+1)} \beta_1^{(m+1)} \cdots \beta_{n-1}^{(m+1)}}{p_{n+1}^{(m)}(z) p_n^{(m+1)}(z)}$$

$$E_{n+1/2}^{(m)}(z) - E_{n-1/2}^{(m+1)}(z) = \frac{1}{E_n^{(m+1)}(z) - E_n^{(m)}(z)}$$

$$= \frac{\begin{vmatrix} 1 & z & \cdots & z^n \\ c_m & c_{m+1} & \cdots & c_{m+n} \\ \vdots & \vdots & & \vdots \\ c_{m+n-1} & c_{m+n} & \cdots & c_{m+2n-1} \end{vmatrix} \begin{vmatrix} 1 & z & \cdots & z^n \\ c_{m+1} & c_{m+2} & \cdots & c_{m+n+1} \\ \vdots & \vdots & & \vdots \\ c_{m+n} & c_{m+n+1} & \cdots & c_{m+2n} \end{vmatrix}}{\begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+n} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+n+1} \\ \vdots & \vdots & & \vdots \\ c_{m+n} & c_{m+n+1} & \cdots & c_{m+2n} \end{vmatrix} \begin{vmatrix} c_{m+1} & c_{m+2} & \cdots & c_{m+n} \\ c_{m+2} & c_{m+3} & \cdots & c_{m+n+1} \\ \vdots & \vdots & & \vdots \\ c_{m+n} & c_{m+n+1} & \cdots & c_{m+2n-1} \end{vmatrix}} \tag{3.8.3}$$

$$= \frac{p_n^{(m)}(z) p_n^{(m+1)}(z)}{c_m \prod_{\mu=0}^{n} e_\mu^{(m)} q_\mu^{(m)}} = \frac{p_n^{(m)}(z) p_n^{(m+1)}(z)}{c_m \beta_0^{(m)} \beta_1^{(m)} \cdots \beta_{n-1}^{(m)}}$$

$$E_{n+1/2}^{(m)}(z) = -\frac{\begin{vmatrix} 0 & 1 & z & \cdots & z^n \\ 1 & c_m & c_{m+1} & \cdots & c_{m+n} \\ z & c_{m+1} & c_{m+2} & \cdots & c_{m+n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ z^n & c_{m+n} & c_{m+n+1} & \cdots & c_{m+2n} \end{vmatrix}}{\begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+n} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+n+1} \\ \vdots & \vdots & & \vdots \\ c_{m+n} & c_{m+n+1} & \cdots & c_{m+2n} \end{vmatrix}} z^{m+1} \tag{3.8.4}$$

It is immediately seen that the quantities $E^{(m)}_{n+1/2}(z)$ are, apart from a factor of z^{m+1}, polynomials of the $2n$th degree in z, and may be written

$$E^{(m)}_{n+1/2}(z) = z^{m+1} \sum_{s=0}^{s=2n} \epsilon^{(m)}_{n,s} z^s. \tag{3.8.5}$$

In this instance a large part of the computational effort which, upon the assumption that $E_s^{(m)}(z)$ $s = \frac{1}{2}, 1, \cdots$ is a general rational function of z, would go into the computation of coefficients which are known to be zero, may be avoided.

3.9 *Example: The Incomplete Gamma Function of Large Argument*

Certain aspects of the preceding theory are illustrated by the integral

$$\{\Gamma(\alpha)\}^{-1} \int_0^\infty \frac{t^{\alpha-1}e^{-t}}{z-t}\,dt \sim \frac{1}{z} + \frac{\alpha}{z^2} + \frac{\alpha(\alpha+1)}{z^3} + \cdots$$
$$= \frac{1}{z-\alpha-} \frac{1\cdot\alpha}{2+\alpha-z-} \cdots \frac{r(\alpha+r-1)}{2r+\alpha-z-} \cdots. \tag{3.9.1}$$

Here

$$c_m = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)} \tag{3.9.2}$$

and as is easily verified, the quantities $q_r^{(m)}$, $e_r^{(m)}$ of the q-d algorithm, are given in this case by

$$q_r^{(m)} = \alpha + m + r - 1, \qquad e_r^{(m)} = r; \tag{3.9.3}$$

further in the notation of equation (3.1.6)

$$p_r^{(0)}(z) = L_r^{(\alpha-1)}(z) \tag{3.9.4}$$

where $L_r^{(\alpha)}(z)$, the Laguerre polynomial of order α and degree n satisfies the recursion

$$(n+1)L^{(\alpha)}_{n+1}(z) - (2n+1+\alpha-z)L^{(\alpha)}_n(z) + (n+\alpha)L^{(\alpha)}_{n-1}(z) = 0. \tag{3.9.5}$$

Replacing z by $-z$, (3.9.1) gives the expansion, suitable for large values of the argument, for the incomplete gamma function

$$\Gamma(1-\alpha, z) \sim \frac{1}{z} - \frac{\alpha}{z^2} + \frac{\alpha(\alpha+1)}{z^3} - \cdots$$
$$= \frac{1}{z+\alpha-} \frac{1\cdot\alpha}{z+\alpha+2-} \cdots \frac{r(\alpha+r-1)}{z+\alpha+2r-} \cdots \tag{3.9.6}$$

The initial members of the E-array for this function are given in Table 1. When $\alpha = 1$, (3.9.6) becomes

$$e^z \int_z^\infty \frac{e^{-t}}{t}\,dt \sim -\sum_{r=0}^{\infty} r!(-z)^{-r-1}$$
$$= \frac{1}{z+1-} \frac{1^2}{z+3-} \cdots \frac{r^2}{z+2r+1-} \cdots \tag{3.9.7}$$

Numerical values of the entries in Table I when $\alpha = z = 1$ are displayed in Table 2.

TABLE 1

0			
$-\sum_{r=0}^{0}\alpha(\alpha+1)\cdots(\alpha+r-1)(-z)^{r+1}$	$\dfrac{1}{z+\alpha}$		
$-\sum_{r=0}^{1}\alpha(\alpha+1)\cdots(\alpha+r-1)(-z)^{r+1}$	$\dfrac{z^{-1}\{z+1\}}{z+\alpha+1}$	$\dfrac{z+\alpha+2}{z^2+2(\alpha+1)z+\alpha(\alpha+1)}$	
$-\sum_{r=0}^{2}\alpha(\alpha+1)\cdots(\alpha+r-1)(-z)^{r+1}$	$\dfrac{z^{-2}\{z^2+2z-\alpha\}}{z+\alpha+2}$	$\dfrac{z^{-1}\{z^2+(\alpha+4)z+2\}}{z^2+2(\alpha+2)z+(\alpha+1)(\alpha+2)}$	$\dfrac{z^2+(2\alpha+6)z+(\alpha^2+4\alpha+6)}{z^3+3(\alpha+2)z^2+3(\alpha+1)(\alpha+2)z+\alpha(\alpha+1)(\alpha+2)}$
$-\sum_{r=0}^{3}\alpha(\alpha+1)\cdots(\alpha+r-1)(-z)^{r+1}$	$\dfrac{z^{-3}\{z^3+3z^2-2\alpha z+\alpha(\alpha+1)\}}{z+\alpha+3}$	$\dfrac{z^{-2}\{z^3+(\alpha+6)z^2+6z-2\alpha\}}{z^2+2(\alpha+3)z+(\alpha+2)(\alpha+3)}$	$\dfrac{z^{-1}\{z^3+(2\alpha+9)z^2+(\alpha^2+7\alpha+18)z+6\}}{z^3+3(\alpha+3)z^2+3(\alpha+2)(\alpha+3)z+(\alpha+1)(\alpha+2)(\alpha+3)}$
$-\sum_{r=0}^{4}\alpha(\alpha+1)\cdots(\alpha+r-1)(-z)^{r+1}$	$\dfrac{z^{-4}\{z^4+4z^3-3\alpha z^2+2\alpha(\alpha+1)z-\alpha(\alpha+1)(\alpha+2)\}}{z+\alpha+4}$	$\dfrac{z^{-3}\{z^4+(\alpha+8)z^3+12z^2-6\alpha z+2\alpha(\alpha+1)\}}{z^2+2(\alpha+4)z+(\alpha+3)(\alpha+4)}$	$\dfrac{z^{-2}\{z^4+(2\alpha+12)z^3+(\alpha^2+10\alpha+36)z^2+24z-6\alpha\}}{z^3+3(\alpha+4)z^2+3(\alpha+3)(\alpha+4)z+(\alpha+2)(\alpha+3)(\alpha+4)}$
	$\dfrac{z^3+(3\alpha+12)z^2+(3\alpha^2+19\alpha+36)z+(\alpha^3+7\alpha^2+18\alpha+24)}{z^4+4(\alpha+3)z^3+6(\alpha+2)(\alpha+3)z^2+4(\alpha+1)(\alpha+2)(\alpha+3)z+\alpha(\alpha+1)(\alpha+2)(\alpha+3)}$		
	$\dfrac{z^{-1}\{z^4+(3\alpha+16)z^3+(3\alpha^2+27\alpha+72)z^2+(\alpha^3+11\alpha^2+46\alpha+96)z+24}{z^4+4(\alpha+4)z^3+6(\alpha+3)(\alpha+4)z^2+4(\alpha+2)(\alpha+3)(\alpha+4)z+(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}$		

TABLE 2

0							
1	0.5					$e^1 \int_1^\infty \frac{e^{-t}}{t}\,dt = 0.5963\ 4786$	
0	$0.\dot{6}$	$0.\dot{5}714$	$285\dot{7}$				
+2	0.5	$0.\dot{6}153$	$84\dot{6}2$	0.5882	3529		
−4	0.8	$0.\dot{5}714$	$285\dot{7}$	0.6027	3973	0.5933	0144
+20	0.0	0.7096	7742	0.5882	3529	0.5988	0240

Some further particular cases of (3.9.6) are the Complementary Error Function

$$(3.9.8)\quad e^{z^2}\int_z^\infty e^{-t^2}\,dt = z\left\{\frac{1}{2z^2+1-}\,\frac{1\cdot 2}{2z^2+5-}\cdots\frac{r(r+1)}{2z^2+4r+1-}\cdots\right\}$$

and Dawson's Integral

$$(3.9.9)\quad e^{-z^2}\int_z^\infty e^{t^2}\,dt \sim -\frac{e^{-z^2}\pi^{1/2}i}{2} + z\left\{\frac{1}{2z^2-1-}\,\frac{1\cdot 2}{2z^2-5-}\cdots\frac{r(r+1)}{2z^2-4r-1-}\cdots\right\}.$$

(This continued fraction is of course divergent, since its convergents are all real and the integral is real. However there remains an uncancelled imaginary constituent upon the right hand side of (3.9.9). Nevertheless the continued fraction (3.9.9) exhibits a sort of semi-convergent behavior. It is to be preferred for the computation of Dawson's integral for large values of the argument to the expansion (4.1.12) which is convergent. Indeed the expansion

$$(3.9.10)\quad \frac{1}{2z-}\,\frac{2}{2z-}\,\frac{4}{2z-}\,\frac{6}{2z-}\cdots$$

the even part of which is the expansion (3.9.9), was used by Barkley Rosser for the tabulation of the Error Function for complex argument [16].) The Sine and Cosine Integrals are given by

$$(3.9.11)\quad Ci(z) + isi(z) = -(\cos z - i\sin z)\cdot\left\{\frac{1}{iz+1-}\,\frac{1^2}{iz+2-}\cdots\frac{r^2}{iz+r+1-}\cdots\right\}.$$

The Fresnel Integrals

$$(3.9.12)\quad C(z) = (2\pi)^{-1/2}\int_0^z t^{-1/2}\cos t\,dt, \qquad S(z) = (2\pi)^{-1/2}\int_0^z t^{-1/2}\sin t\,dt$$

may be computed by writing

$$(3.9.13)\quad C(z) = \tfrac{1}{2} - (2\pi)^{-1/2}C(z,\tfrac{1}{2}) \qquad S(z) = \tfrac{1}{2} - (2\pi)^{-1/2}S(z,\tfrac{1}{2})$$

where

$$C(z,\tfrac{1}{2}) - iS(z,\tfrac{1}{2}) = (\sin z - i\cos z)z^{1/2}\left\{\frac{2}{2iz+1-}\,\frac{1\cdot 2}{2iz+5-}\cdots\frac{r(r+1)}{2iz+4r+1-}\cdots\right\}.$$

4. The Padé Table. The remarks contained in §2 indicate that the upper half of the Padé Table may be constructed by methods described for the construction of the E-array.

It would appear, since the system of orthogonal polynomials $p_r^{(m)}(z)$ does not exist for negative values of m, that the use of the results in the theory of orthogonal polynomials as described, for the purpose of constructing the lower half of the Padé table, is not possible. Nevertheless, if the first three polynomials can be determined in some other way, the sequence may be continued by use of the results as described, since the unknown quantities $\beta_0^{(m)}, \beta_1^{(m)}, \cdots, \beta_{-m-1}^{(m)}, \alpha_0^{(m)}, \alpha_1^{(m)}, \cdots, \alpha_{-m}^{(m)}$ where m is negative, disappear from the resulting computations.

In terms of q-d scheme, as defined by equations (3.3.7), (3.3.8), the rational expressions

$$\frac{U_{0,m-1}(x)}{V_{0,m-1}(x)}, \quad \frac{U_{0,m}(x)}{V_{0,m}(x)}, \quad \frac{U_{1,m}(x)}{V_{1,m}(x)}, \quad \frac{U_{1,m+1}(x)}{V_{1,m+1}(x)}, \cdots$$

are the successive convergents of the continued fraction

$$\sum_{s=0}^{m-1} c_s x^s + \frac{c_m x^m}{1-} \frac{q_1^{(m)} x}{1-} \frac{e_1^{(m)} x}{1-} \frac{q_2^{(m)} x}{1-} \frac{e_2^{(m)} x}{1-} \cdots \tag{4.1}$$

while the rational expressions

$$\frac{U_{0,m-1}(x)}{V_{0,m-1}(x)}, \quad \frac{U_{1,m}(x)}{V_{1,m}(x)}, \quad \frac{U_{2,m+1}(x)}{V_{2,m+1}(x)}, \cdots$$

are the successive convergents of the even part of expansion (4.1), namely,

$$\sum_{s=0}^{m-1} c_s x^s + \frac{c_m x^m}{1 - q_1^{(m)} x -} \frac{e_1^{(m)} q_1^{(m)} x^2}{1 - (q_2^{(m)} + e_1^{(m)}) x -} \cdots \frac{e_r^{(m)} q_r^{(m)} x^2}{1 - (q_{r+1}^{(m)} + e_r^{(m)}) x -} \cdots . \tag{4.2}$$

If the series $\sum_{s=0}^{\infty} d_s x^s$ is derived by means of the relation

$$\left(\sum_{s=0}^{\infty} c_s x^s\right)\left(\sum_{s=0}^{\infty} d_s x^s\right) = 1 \tag{4.3}$$

and the quantities derived by applying the relationships of the q-d algorithm to the quantities d_s, $s = 0, 1, \cdots$, are denoted by $q_r^{(m)\prime}$, $e_r^{(m)\prime}$, then the rational functions

$$\frac{U_{m-1,0}(x)}{V_{m-1,0}(x)}, \quad \frac{U_{m,0}(x)}{V_{m,0}(x)}, \quad \frac{U_{m,1}(x)}{V_{m,1}(x)}, \quad \frac{U_{m+1,1}(x)}{V_{m+1,1}(x)}, \cdots$$

are the successive convergents of the continued fraction

$$\frac{1}{d_0 + d_1 x + d_2 x^2 + \cdots + d_{m-1} x^{m-1}} + \frac{d_m x^m}{1-} \frac{q_1^{(m)\prime} x}{1-} \frac{e_1^{(m)\prime} x}{1-} \frac{q_2^{(m)\prime} x}{1-} \frac{e_2^{(m)\prime} x}{1-} \cdots \tag{4.4}$$

If the quantities d_s, $s = 0, 1, \cdots$, defined by equation (4.3) are required in isolation, they are most efficiently derived by solving the scheme

$$\sum_{s=0}^{s=r} d_s c_{r-s} = \begin{cases} 1 & r = 0 \\ 0 & r \neq 0 \end{cases}$$

If, however, the quantities $q_r^{(m)'}$, $e_r^{(m)'}$ are also required, some economy in the computation may be achieved by use of a result in the theory of continued fractions [16], [2, p. 332], namely that if

$$\beta(x) = \frac{c_0}{1-}\frac{q_1^{(0)}x}{1-}\frac{e_1^{(0)}x}{1-}\frac{q_2^{(0)}x}{1-}\cdots \tag{4.5}$$

and

$$\{\beta(x)\}^{-1} = \frac{d_0}{1-}\frac{q_1^{(0)'}x}{1-}\frac{e_1^{(0)'}x}{1-}\frac{q_2^{(0)'}x}{1-}\frac{e_2^{(0)'}x}{1-}\cdots, \tag{4.6}$$

then

$$\begin{aligned} &d_0 = c_0^{-1}, \qquad q_1^{(0)'} = -q_1^{(0)}, \qquad e_1^{(0)'} = q_1^{(0)} + e_1^{(0)}, \\ &e_r^{(0)'}q_{r+1}^{(0)'} = e_r^{(0)}q_{r+1}^{(0)}, \qquad e_{r+1}^{(0)'} + q_{r+1}^{(0)'} = e_{r+1}^{(0)} + q_{r+1}^{(0)}, \qquad r \geqq 1 \end{aligned} \tag{4.7}$$

In the determination of the quantities $q_r^{(m)}$, $e_r^{(m)}$ the q-d scheme is built up from left to right by means of the q-d relationships (3.3.7) and (3.3.8), while in the determination of the quantities $q_r^{(m)'}$, $e_r^{(m)'}$, the diagonal $q_1^{(0)'}$, $e^{(0)'}$, $q_2^{(0)'}$, $e_2^{(0)'}$, $\cdots$ is determined by equations (4.7), and the q-d relationships are used to build up the q-d scheme from right to left.

It is to be expected that the recursion formulas between the systems of functions $r_n^{(m)}(z)$, $p_n^{(m)}(z)$ are complemented by recursions between the polynomials $U_{\mu,\nu}(x)$, $V_{\mu,\nu}(x)$ occurring in the Padé table. Indeed Frobenius [17] has given such a system of recursion formulas. Adopting his notation, write

$$c_{\mu,\nu} = \begin{vmatrix} c_{\mu-\nu+1} & c_{\mu-\nu+2} & \cdots & c_{\mu} \\ c_{\mu-\nu+2} & c_{\mu-\nu+3} & \cdots & c_{\mu+1} \\ \vdots & \vdots & & \vdots \\ c_{\mu} & c_{\mu+1} & \cdots & c_{\mu+\nu-1} \end{vmatrix} \tag{4.8}$$

$$T_{\mu,\nu} = \begin{vmatrix} c_{\mu-\nu+1} & c_{\mu-\nu+2} & \cdots & c_{\mu} & c_{\mu-\nu}x^{\mu} + c_{\mu-\nu-1}x^{\mu-1} + \cdots \\ c_{\mu-\nu+2} & c_{\mu-\nu+3} & \cdots & c_{\mu+1} & c_{\mu-\nu+1}x^{\mu} + c_{\mu-\nu}x^{\mu-1} + \cdots \\ \vdots & \vdots & & \vdots & \vdots \\ c_{\mu+1} & c_{\mu+2} & \cdots & c_{\mu+\nu} & c_{\mu}x^{\mu} + c_{\mu-1}x^{\mu-1} + \cdots \end{vmatrix} \tag{4.9}$$

$$S_{\mu,\nu} = \begin{vmatrix} c_{\mu-\nu+1} & c_{\mu-\nu+2} & \cdots & c_{\mu} & x^{\nu} \\ c_{\mu-\nu+2} & c_{\mu-\nu+3} & \cdots & c_{\mu+1} & x^{\nu-1} \\ \vdots & \vdots & & \vdots & \vdots \\ c_{\mu+1} & c_{\mu+2} & \cdots & c_{\mu+\nu} & 1 \end{vmatrix} \tag{4.10}$$

so that

$$c_{\mu,\nu} = H_{\mu-\nu+1}^{(\nu)} \tag{4.11}$$

$$\beta(x)S_{\mu,\nu} - T_{\mu,\nu} = c_{\mu+1,\nu+1}\,x^{\mu+\nu+1} + \cdots \tag{4.12}$$

$$T_{\mu,\nu} = U_{\mu,\nu}(x)c_{\mu+1,\nu} \tag{4.13}$$

$$S_{\mu,\nu} = V_{\mu,\nu}(x)c_{\mu+1,\nu}\,. \tag{4.14}$$

He then gives the following system of homogeneous recursion formulas, where $R_{\mu,\nu}$ is taken to mean either $S_{\mu,\nu}$ or $T_{\mu,\nu}$

$$c_{\mu+1,\nu}R_{\mu-1,\nu} - c_{\mu,\nu+1}R_{\mu,\nu-1} = c_{\mu,\nu}R_{\mu,\nu} \tag{4.15}$$

$$c_{\mu,\nu+1}R_{\mu+1,\nu} - c_{\mu+1,\nu}R_{\mu,\nu+1} = c_{\mu+1,\nu+1}xR_{\mu,\nu} \tag{4.16}$$

$$c_{\mu+1,\nu}R_{\mu,\nu} - c_{\mu,\nu}R_{\mu+1,\nu} = c_{\mu+1,\nu+1}xR_{\mu,\nu-1} \tag{4.17}$$

$$c_{\mu,\nu+1}R_{\mu,\nu} - c_{\mu,\nu}R_{\mu,\nu+1} = c_{\mu+1,\nu+1}xR_{\mu-1,\nu} \tag{4.18}$$

$$c_{\mu,\nu}c_{\mu,\nu+1}R_{\mu+1,\nu} - (c_{\mu,\nu+1}c_{\mu+1,\nu} + c_{\mu,\nu}c_{\mu+1,\nu+1}x) \cdot R_{\mu,\nu} + c_{\mu+1,\nu}c_{\mu+1,\nu+1}xR_{\mu-1,\nu} = 0 \tag{4.19}$$

$$c_{\mu,\nu}c_{\mu+1,\nu}R_{\mu,\nu+1} - (c_{\mu,\nu+1}c_{\mu+1,\nu} - c_{\mu,\nu}c_{\mu+1,\nu+1}x) \cdot R_{\mu,\nu} + c_{\mu,\nu+1}c_{\mu+1,\nu+1}xR_{\mu,\nu-1} = 0 \tag{4.20}$$

$$c_{\mu,\nu}^2R_{\mu+1,\nu+1} - \{c_{\mu,\nu}c_{\mu+1,\nu+1} + (c_{\mu+1,\nu+2}c_{\mu,\nu-1} - c_{\mu+2,\nu+2}c_{\mu-1,\nu})x\} \cdot R_{\mu,\nu} + c_{\mu+1,\nu+1}^2x^2R_{\mu-1,\nu-1} = 0. \tag{4.21}$$

Equation (4.19) is a three term linear homogeneous recursion between the numerators and denominators of constituents of the Padé table lying in the same row, equation (4.20) another such relating to a column. Equation (4.21) relates numerators and denominators of constituents in the Padé table lying along a diagonal.

It will immediately be realized that the triplets of equations (3.4.1), (3.4.5) and (4.17); (3.4.2), (3.4.6) and (4.18); (3.4.8), (3.4.10) and (4.19); (3.4.7), (3.4.9) and (4.20); (3.1.16), (3.1.18), and (4.21) are identical. Frobenius also gives a further system of inhomogeneous recursions, which do not seem to have immediate practical application. The function

$$\frac{U_{0,m-1}(x)}{V_{0,m-1}(x)}, \qquad \frac{U_{0,m}(x)}{V_{0,m}(x)}, \qquad \frac{U_{1,m}(x)}{V_{1,m}(x)}, \cdots$$

may also be formed by applying the Euler-Minding relations to the continued fraction

$$\sum_{s=0}^{m-1} c_s x^s + \frac{c_m x^m}{1-}\,\frac{q_1^{(m)}x}{1-}\,\frac{e_1^{(m)}x}{1-} \cdots \frac{q_r^{(m)}x}{1-}\,\frac{e_r^{(m)}x}{1-} \cdots \tag{4.22}$$

in direct analogy to the process preceding equation (3.5.4): further, the functions

$$\frac{U_{r,m-1+r}(x)}{V_{r,m-1+r}(x)} = \sum_{s=0}^{m-1} c_s x^s + \frac{c_m x^m}{1-}\,\frac{q_1^{(m)}x}{1-}\,\frac{e_1^{(m)}x}{1-} \cdots \frac{q_r^{(m)}x}{1} \tag{4.23}$$

and

$$\frac{U_{r,m+r}(x)}{V_{r,m+r}(x)} = \sum_{s=0}^{m-1} c_s x^s + \frac{c_m x^m}{1-}\,\frac{q_1^{(m)}x}{1-}\,\frac{e_1^{(m)}x}{1-} \cdots \frac{q_r^{(m)}x}{1-}\,\frac{e_r^{(m)}x}{1} \tag{4.24}$$

may be built up from the non-linear recursions

$$C_n = b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+} \cdots \frac{a_n}{b_n} \tag{4.25}$$

$$D_0 = b_n, \qquad D_r = b_{n-r} + \frac{a_{n-r+1}}{D_{r-1}}, \qquad D_n = C_n. \tag{4.26}$$

The same remarks apply, of course, to the continued fraction

$$\sum_{s=0}^{m-1} d_s x^s + \frac{d_m x^m}{1-} \frac{q_1^{(m)'} x}{1-} \frac{e_1^{(m)'} x}{1-} \cdots \frac{q_r^{(m)'} x}{1-} \frac{e_r^{(m)'} x}{1-} \cdots \tag{4.27}$$

the functions computed from (4.27) being the sequence

$$\frac{V_{m-1,0}(x)}{U_{m-1,0}(x)}, \quad \frac{V_{m,0}(x)}{U_{m,0}(x)}, \quad \frac{V_{m,1}(x)}{U_{m,1}(x)}, \cdots.$$

The ϵ-algorithm may also be used to construct the Padé table: the partial sums $\sum_{s=0}^{r} c_s x^s \quad r = 0, 1, \cdots$ are placed in the leading column, the quantities $(c_r x^r)^{-1}$ $r = 1, 2, \cdots$ in the second, and the ϵ-algorithm relationships (3.8.1) applied. The functions

$$\frac{U_{0,s}(x)}{V_{0,s}(x)}, \quad \frac{U_{1,s}(x)}{V_{1,s}(x)}, \quad \frac{U_{2,s}(x)}{V_{2,s}(x)}, \cdots$$

then appear in the first, third, fifth and further columns respectively; the array must thus be transposed about the leading diagonal if the notation employed in the description of the Padé table is correctly to be observed. The remainder of the Padé table may be computed by applying the same process to the series $\sum_{s=0}^{\infty} d_s x^s$ it being necessary subsequently to invert the entries in the first, third, fifth and further columns which remain however in their correct position. The entries in the principal diagonal will in this way have been computed twice, the two sets of results should of course agree.

It is not necessary to compute the partial sums of the inverse series $\sum_{s=0}^{\infty} d_s x^s$, defined by (2.12), in order to construct the whole Padé table by means of the ϵ-algorithm. In the notation of the ϵ-array one adds the boundary values $\epsilon_{2s}^{(-s-1)} = 0$, and this enables both halves of the Padé table to be constructed. (See P. Wynn, "L'ϵ-algoritmo e la tavola Padé," to appear.)

4.1 *Example*: *The Incomplete Beta Function*

An example which may be used to illustrate the theory relating to the Padé table is provided by the series

$${}_2F_1(1, a; c; x) = 1 + \frac{a}{c} x + \frac{a(a+1)x^2}{c(c+1)} + \cdots. \tag{4.1.1}$$

The initial members of the Padé table for this function are given in Table 3.

TABLE 3

1	$1 + \frac{a}{c} x$	$1 + \frac{a}{c} x + \frac{a(a+1)}{c(c+1)} x^2$
$\frac{1}{1 - \frac{a}{c} x}$	$\frac{1 + \frac{1(a-c)}{c(c+1)} x}{1 - \frac{(a+1)}{(c+1)} x}$	$\frac{1 + \frac{2(a-c)}{c(c+2)} x + \frac{1a(a-c)}{c(c+1)(c+2)} x^2}{1 - \frac{(a+2)}{(c+2)} x}$
$\frac{1}{1 - \frac{a}{c} x + \frac{a(a-c)}{c^2(c+1)} x^2}$	$\frac{1 + \left\{\frac{(a-c)(c+2) - (a+1)c^2}{c(c+1)(c+2)}\right\} x}{1 - \frac{2(a+1)}{(c+2)} x + \frac{a(a+1)}{(c+1)(c+2)} x^2}$	$\frac{1 + \left\{\frac{a}{c} - \frac{2(a+2)}{(c+3)}\right\} x + \frac{2(a-c)(a-c-1)x^2}{c(c+1)(c+2)(c+3)}}{1 - \frac{2(a+2)}{(c+3)} x + \frac{(a+1)(a+2)}{(c+2)(c+3)} x^2}$

As is easily verified, the q-d tableau relating to the series (4.1.1) is given in Table 4.

TABLE 4

$q_1^{(m)}$	$e_1^{(m)}$	$q_2^{(m)}$	$q_r^{(m)}$	$e_r^{(m)}$
$\frac{a}{c}$				
	$\frac{1(c-a)}{c(c+1)}$			
$\frac{(a+1)}{(c+2)}$		$\frac{(a+1)c}{(c+1)(c+2)}$		
	$\frac{1(c-a)}{(c+1)(c+2)}$			
$\frac{(a+2)}{(c+2)}$		$\frac{(a+2)(c+1)}{(c+2)(c+3)}$	$\frac{(a+r-1)(c+r-2)}{(c+2r-3)(c+2r-2)}$	
	$\frac{1(c-a)}{(c+2)(c+3)}$			$\frac{r(c-a+r-1)}{(c+2r-2)(c+2r-1)}$ ⋱
$\frac{(a+3)}{(c+3)}$	⋮	$\frac{(a+3)(c+2)}{(c+3)(c+4)}$	$\frac{(a+r)(c+r-1)}{(c+2r-2)(c+2r-1)}$	
⋮		⋮	⋮	$\frac{r(c-a+r-1)}{(c+2r-1)(c+2r)}$ ⋱
				⋮

More generally

$$q_r^{(m)} = \frac{(a+r+m-1)(c+r+m-2)}{(c+2r+m-3)(c+2r+m-2)}, \tag{4.1.2}$$

$$e_r^{(m)} = \frac{r(c-a+r-1)}{(c+2r+m-2)(c+2r+m-1)}$$

from which follows the continued fraction expansion

$$1+\sum_{r=1}^{\infty}\frac{a(a+1)\cdots(a+r-1)}{c(c+1)\cdots(c+r-1)}x^r = 1+\sum_{r=1}^{m}\frac{a(a+1)\cdots(a+r-1)}{c(c+1)\cdots(c+r-1)}x^r$$

$$+\frac{a(a+1)\cdots(a+m)}{c(c+1)\cdots(c+m)}x^{m+1}\left\{\frac{1}{1-}\,\frac{(a+m)x}{c+m-}\,\frac{1(c-a)x}{c+m+1-}\right. \tag{4.1.3}$$

$$\left.\cdots\frac{(a+m+r)(c+m+r-1)x}{c+m+2r-}\,\frac{(r+1)(c-a+r)x}{c+m+2r+1-}\cdots\right\}.$$

The continued fraction

$$\frac{1}{1-}\,\frac{ax}{c-}\,\frac{1(c-a)x}{c+1-}\,\frac{(a+1)cx}{c+2-}\,\frac{2(c-a+1)x}{c+3-} \tag{4.1.4}$$

$$\cdots\frac{(a+r)(c+r-1)x}{c+2r+2-}\,\frac{(r+1)(c-a+r)x}{c+2r+2-}\cdots$$

which corresponds to (4.1.3) with $m = 0$, was used by Müller [18] in the preparation of a short table of the Incomplete Beta Function. When $a = 1$, $c = 2$, and x is replaced by $-x$, (4.1.4) reduces to the well known expansion

$$\log(1+x) = \frac{x}{1+}\,\frac{1^2x}{2+}\,\frac{1^2x}{3+}\,\frac{2^2x}{4+}\cdots\frac{r^2x}{2r+}\,\frac{r^2x}{2r+1+}\cdots. \tag{4.1.5}$$

Table 3 becomes in this case (numerical values for $x = 1$ are appended) as shown in Table 5.

TABLE 5

1	$1 - \frac{1}{2}x$	$1 - \frac{1}{2}x + \frac{1}{12}x^2$	1.0	0.5	0.833333
$\frac{1}{1 + \frac{1}{2}x}$	$\frac{1 + \frac{1}{6}x}{1 + \frac{2}{3}x}$	$\frac{1 + \frac{1}{4}x - \frac{1}{24}x^2}{1 + \frac{3}{4}x}$	0.6	0.7	0.690476
$\frac{1}{1 + \frac{1}{2}x + \frac{1}{12}x^2}$	$\frac{1 + \frac{1}{2}x}{1 + x + \frac{1}{6}x^2}$	$\frac{1 + \frac{7}{10}x + \frac{1}{10}x^2}{1 + \frac{6}{5}x + \frac{3}{10}x^2}$	0.705882	0.692308	0.693333
			$\log_e 2 = 0.693147\cdots$		

When $a = \frac{1}{2}$, $c = \frac{3}{2}$ and x is replaced by $-x^2$, there follows

$$\text{(4.1.6)}\quad \arctan x = \frac{x}{1+}\,\frac{1^2x^2}{3+}\,\frac{4\cdot 1^2x^2}{5+}\,\frac{3^2x^2}{7+}\,\frac{4\cdot 2^2x^2}{9+}\cdots\frac{(2r+1)^2x^2}{4r+3+}\,\frac{4(r+1)^2x^2}{4r+5+}\cdots.$$

Replacing x by x/a in (4.1.4) and letting a tend to infinity, there follows the continued fraction expansion.

$$\text{(4.1.7)}\quad \frac{1}{1-}\,\frac{x}{c+}\,\frac{1x}{c+1-}\,\frac{cx}{c+2+}\,\frac{2x}{c+3-}\cdots\frac{rx}{c+2r-1-}\,\frac{(c+r-1)x}{c+2r+}\cdots$$

of the incomplete gamma function

$$cx^{-c}e^x\int_0^x e^{-t}t^{c-1}\,dt$$

suitable for small values of x. The initial members of the Padé table for this series are shown in Table 6.

TABLE 6

1	1	$1 + \frac{x}{c} + \frac{x^2}{c(c+1)}$
$\frac{1}{1 - \frac{x}{c}}$	$\frac{1 + \frac{x}{(c+1)}}{1 - \frac{x}{(c+1)}}$	$\frac{1 + \frac{2}{c(c+2)}x + \frac{x^2}{c(c+1)(c+2)}}{1 - \frac{x}{(c+2)}}$
$\frac{1}{1 - \frac{x}{c} + \frac{x^2}{c^2(c+1)}}$	$\frac{1 - \frac{(c-2)}{c(c+2)}x}{1 - \frac{2}{(c+2)}x + \frac{x^2}{(c+1)(c+2)}}$	$\frac{1 - \frac{(c-3)}{c(c+3)}x + \frac{2}{c(c+1)(c+2)(c+3)}x^2}{1 - \frac{2}{(c+3)}x + \frac{x^2}{(c+2)(c+3)}}$

When $c = \frac{3}{2}$, and x is replaced by x^2 in (4.1.7), there follows the continued fraction expansion

$$\text{(4.1.8)}\quad xe^{-x^2}\left\{\frac{1}{1-}\,\frac{2x^2}{3+}\,\frac{4x^2}{5-}\,\frac{6x^2}{7+}\cdots\right\}$$

of the error function

$$\operatorname{erf}(x) = \int_0^x e^{-t^2}\,dt$$

due to Laplace [19], which was used by Sheppard for the computation of the British Association tables [20] of the error function. When $c = 1$, (4.1.7) gives the well known expansion of the exponential function

$$(4.1.9)\quad \exp(x) = 1 + \frac{x}{1-}\,\frac{x}{2+}\,\frac{x}{3-}\,\frac{2x}{4+}\,\frac{2x}{5-}\cdots\frac{rx}{2r+}\,\frac{rx}{2r+1-}\cdots$$

and Table 6 is replaced by (numerical values for $x = 1$ being appended) Table 6a.

TABLE 6a

1	$1+x$	$1+x+\frac{x^2}{2}$	1.0	2.0	2.5
$\frac{1}{1-x}$	$\frac{1+\frac{x}{2}}{1-\frac{x}{2}}$	$\frac{1+\frac{2}{3}x+\frac{x^2}{12}}{1-\frac{x}{3}}$	∞	3.0	2.75
$\frac{1}{1-x+\frac{x^2}{2}}$	$\frac{1+\frac{x}{3}}{1-\frac{2}{3}x+\frac{x^2}{6}}$	$\frac{1+\frac{x}{2}+\frac{x^2}{12}}{1-\frac{x}{2}+\frac{x^2}{12}}$	2.0	2.6	2.714 286
			$\exp(1) = 2.718\,282$		

By replacing $-x^2$ by ix in (4.1.8) the Fresnel integrals

$$(4.1.10)\quad C(x) = (2\pi)^{-1/2}\int_0^x t^{-1/2}\cos t\,dt \quad\text{and}\quad S(x) = (2\pi)^{-1/2}\int_0^x t^{-1/2}\sin t\,dt$$

may be computed at one blow by evaluating the real and imaginary parts of the expansion

$$(4.1.11)\quad C(x) + iS(x) = \left(\frac{2}{\pi}\right)^{1/2} x^{1/2}(\cos x + i\sin x)\left\{\frac{1}{1+}\,\frac{2ix}{3-}\,\frac{4ix}{5+}\,\frac{6ix}{7-}\cdots\right\}$$

Further, by writing x^2 for $-x^2$ in (4.1.8) there follows the expansion of Dawson's integral

$$(4.1.12)\qquad \int_0^x e^{t^2}\,dt = xe^{x^2}\left\{\frac{1}{1+}\,\frac{2x^2}{3-}\,\frac{4x^2}{5+}\,\frac{6x^2}{7-}\cdots\right\}.$$

Explicit formulas for the rational approximations corresponding to the functions (3.9.6), (4.1.4) and (4.1.7) are given by Luke [21].

5. Series with Hadamard Gaps. It has been tacitly assumed throughout that none of the Hankel determinants $H_r^{(k)}$, defined by equation (2–4), involving the coefficients in the power series

$$\beta(x) = \sum_{s=0}^{\infty} c_s x^s \quad\text{and}\quad F(z) = \sum_{s=0}^{\infty} c_s z^{-s-1},$$

vanishes. If this condition is not fulfilled, certain formal difficulties exist in the methods described in this paper, though straight-forward appeal to the set of equations (2.3) may be sufficient to determine certain of the rational quotients. A feature of the Padé table derived for series of this kind is that there are blocks containing the rational functions lying in and upon the rectangle at whose corners stand the quotients

$$\frac{U_{\mu,\nu}(x)}{V_{\mu,\nu}(x)}, \quad \frac{U_{\mu+m,\nu}(x)}{V_{\mu+m,\nu}(x)}, \quad \frac{U_{\mu,\nu+n}(x)}{V_{\mu,\nu+n}(x)}, \quad \frac{U_{\mu+m,\nu+n}(x)}{V_{\mu+m,\nu+n}(x)}$$

which, due to cancellation of polynomial factors, contain only the quotient $U_{\mu,\nu}(x)/V_{\mu,\nu}(x)$. The continued fractions whose successive convergents are the distinct quotients lying upon a diagonal in the Padé table are of the form

$$c_0 + \frac{a_1 x^{\alpha_1}}{1+} \frac{a_2 x^{\alpha_2}}{1+} \cdots \tag{4.5.1}$$

where $\alpha_1, \alpha_2, \cdots$ are positive integers. A general theory of such continued fractions and of the resulting rational approximations is not yet available, though Wall [3, p. 399] gives a comprehensive survey of work which has been done upon this problem, and makes a number of interesting conjectures.

It is to be expected that numerical experience obtained in the use of rational approximation formulas will point the way to further theoretical developments.

6. The Efficiency of Various Methods of Obtaining Rational Approximations. The process of obtaining rational approximations to a function formally defined by a power series expansion may be regarded as one solution to the problem of obtaining information about the functions

$$\beta(x) = \sum_{s=0}^{\infty} c_s x^s \quad \text{or} \quad F(z) = \sum_{s=0}^{\infty} c_s z^{-s-1}$$

from the finite set of coefficients $c_s \quad s = 0, 1, \cdots, \nu$, where, for convenience in the ensuing discussion, ν is taken to be $2n - 1$. Numerical experience supports the assertion, as does reference to Tables 3, 5 and 7 in the case of particular examples, that in general, of those rational expressions $U_{\mu,\nu}(x)/V_{\mu,\nu}(x)$ or $E_r^{(m)}(z)$ which may be obtained from the set of coefficients $c_s \quad s = 0, 1, \cdots, 2n - 1$, (namely those entries in the Padé table lying upon and in the triangle

$$\frac{U_{0,\nu}(x)}{V_{0,\nu}(x)}, \qquad \nu = 0, 1, \cdots, 2n - 1,$$

$$\frac{U_{\mu,0}(x)}{V_{\mu,0}(x)} \qquad \mu = 0, 1, \cdots, 2n - 1, \qquad\qquad \frac{U_{2n-\nu-1,\nu}(x)}{V_{2n-\nu-1,\nu}(x)} \qquad \nu = 0, 1, \cdots, 2n - 1,$$

and those entries in the E-array which lie in and upon the triangle whose vertices coincide with the functions $E_0^{(0)}(z)$, $E_n^{(0)}(z)$, $E_0^{(2n)}(z)$,) the expressions which contain the most information about the function $\beta(x)$ or $F(z)$, or more precisely, for prescribed x or z the expressions for which

$$\left| \frac{U_{\mu,\nu}(x)}{V_{\mu,\nu}(x)} - \beta(x) \right| \quad \text{or} \quad |F(z) - E_r^{(m)}(z)|$$

are a minimum, are given by $\mu + 1 = \nu = n$ or $\mu = \nu + 1 = n$, or by $m = 0$, $r = n$.

A comparison of the methods devised from results contained in the theoretical part of this paper, which is meaningful in a practical context, may thus be made by contrasting the numbers and types of arithmetic operations needed to compute (versing the discussion in terms of the E-array for convenience)

I. the function $\dfrac{o_r^{(0)}(z)}{p_r^{(0)}(z)}$

II. the sequence of functions $\dfrac{o_r^{(0)}(z)}{p_r^{(0)}(z)} \quad r = 1, \cdots, n$

III. the complete array of functions $\dfrac{r_r^{(m)}(z)}{p_r^{(m)}(z)} \quad r = 1, 2, \cdots, n;$

$$m = 0, \cdots, 2n - 2r$$

The remarks contained in sections 2 and 3 indicate that in general any conclusions arrived at relating to methods for the construction of the E-array apply with equal force to the construction of the Padé table.

In this comparison, which is about to be made, a further distinction will be drawn between the processes of deriving explicit formulas for the rational functions, and that of computing the values of these functions for some prescribed values of the argument, as would be required, for example, in an initial exploratory study. It is quite clear that in many cases the techniques employed in these two processes will differ, and that considerable saving of computational effort may follow from the foreknowledge that numerical values of the various expressions for a single value of the argument are to be required. For example, assuming $\alpha_r^{(m)}$, $\beta_{r-1}^{(m)}$ to be given, the computation of the coefficients in the polynomial $p_{r+1}^{(m)}(z)$ from the recursion

$$p_{r+1}^{(m)}(z) = (z - \alpha_r^{(m)})p_r^{(m)}(z) - \beta_{r-1}^{(m)}(z) \tag{6.1}$$

requires $2r - 1$ multiplications and $2r - 1$ subtractions, whereas if the polynomials $p_s^{(m)}(z)$ are regarded as numerical quantities, the recursion (6.1) involves merely 2 subtractions and 2 multiplications. A distinction of a more subtle kind is indicated by the two recursions

$$p_r^{(m)}(z) = zp_{r-1}^{(m+1)} - q_r^{(m)}p_{r-1}^{(m)}(z) \tag{6.2}$$

$$p_r^{(m+1)}(z) = p_r^{(m)}(z) - e_r^{(m)}p_{r-1}^{(m+1)}(z). \tag{6.3}$$

For numerical values of z, the second is to be preferred to the first for the computation of the polynomials $p_r^{(m)}(z)$ since only one multiplication is required (the polynomials $p_r^{(0)}(z) \quad r = 1, 2, \cdots, n$ must, of course, be computed previously to the computation of $p_r^{(m)}(z)$, either by use of (6.1) with $m = 0$, or by judiciously timed application of (6.2) with $m = 0$). With regard to the computation of the coefficients in the polynomial however, the first is to be preferred, since multiplication by z corresponds only to the shift of the coefficients vector

$$(k_{r-1,r-1}^{(m+1)}, k_{r-1,r-2}^{(m+1)}, \cdots, k_{r-1,0}^{(m+1)})$$

and the recursion itself is applicable throughout the whole range of m and r under consideration.

Any feasible method for constructing the rational functions $E_r^{(m)}(z)$ directly from the coefficients $c_s \quad s = 0, 1, \cdots$ may be regarded as proceeding in a number of stages, each of which incorporates a result described earlier in this paper. A systematic study of feasible methods for deriving rational approximations may thus be conducted by describing each of these stages in terms of the numbers and types of arithmetic operations involved and, subsequently, designing the methods by combining suitable steps. The arithmetic operations involved in each method may be assessed by adding together those involved in the composite stages, and a comparison of the methods thus be made.

The following notations will be used:

for the vectors:

$$\mathbf{p}_r^{(m)} = (k_{r,r}^{(m)}, k_{r,r-1}^{(m)}, \cdots, k_{r,0}^{(m)}) \tag{6.4}$$

$$\mathbf{o}_r^{(m)} = (j_{r,r-1}^{(m)}, j_{r,r-2}^{(m)}, \cdots, j_{r,0}^{(m)}) \tag{6.5}$$

$$\mathbf{r}_r^{(m)} = (i_{r,m+r-1}^{(m)}, i_{r,m+r-2}^{(m)}, \cdots, i_{r,0}^{(m)}) \tag{6.6}$$

$$\mathbf{c}_r^{(m)} = (c_m, c_{m+1}, \cdots, c_{m+r}) \tag{6.7}$$

$$\mathbf{d}_{t,s}^{(m)} = (d_{t,s,s-1}^{(m)}, d_{t,s,s-2}^{(m)}, \cdots, d_{t,s,0}^{(m)}) \tag{6.8}$$

$$\mathbf{d}_{t,s}^{(m)'} = (d_{t,s,s}^{(m)'}, d_{t,s,s-1}^{(m)'}, \cdots, d_{t,s,0}^{(m)'}) \tag{6.9}$$

$$\mathbf{e}_r^{(m)} = (\epsilon_{r,2r}^{(m)}, \epsilon_{r,2r-1}^{(m)}, \cdots, \epsilon_{r,0}^{(m)}) \tag{6.10}$$

$$\mathbf{z}_r^{(m)} = (z^m, z^{m+1}, \cdots, z^{m+r}) \tag{6.11}$$

for the matrices:

$$\mathbf{C}_r^{(m)} = \begin{pmatrix} c_m & c_{m+1} & \cdots & c_{m+r} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+r+1} \\ \vdots & \vdots & & \vdots \\ c_{m+r} & c_{m+r+1} & \cdots & c_{m+2r} \end{pmatrix} \tag{6.12}$$

$$\mathbf{H}_r^{(m)} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ h_{1,0}^{(m)} & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ h_{r,0}^{(m)} & h_{r,1}^{(m)} & \cdots & 1 \end{pmatrix} \tag{6.13}$$

$$\mathbf{K}_r^{(m)} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ k_{1,0}^{(m)} & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ k_{r,0}^{(m)} & k_{r,1}^{(m)} & \cdots & 1 \end{pmatrix} \tag{6.14}$$

The matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots \\ 0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{6.15}$$

possessing i rows and j columns, derived from the vector $\mathbf{a}_k = (a_1, a_2, \cdots, a_k)$ where $a_s = 0$, $s > k$, will be denoted by $[\mathbf{a}_k]_{i,j}$. The extended vectors $(0, \cdots$.

$a_1, a_2, \cdots, a_k$) and $(a_1, a_2, \cdots, a_k, 0, \cdots, 0)$ each possessing t elements, will be denoted by $(0, \cdots, 0, \mathbf{a}_k)_t$ and $(\mathbf{a}_k, 0, \cdots, 0)_t$ respectively.
The stages are as follows:

I. Formation of the quantities $q_r^{(m)}$ $r = 1, 2, \cdots, n; m = 0, 1, \cdots, 2n - 2r$, $e_r^{(m)}$ $r = 0, 1, \cdots, n - 1; m = 0, 1, \cdots, 2n - 2r - 1$, by means of the relations

$$e_0^{(m)} = 0 \qquad q_1^{(m)} = c_{m+1}/c_m$$
$$q_r^{(m)} + e_r^{(m)} = q_r^{(m+1)} + e_{r-1}^{(m+1)}$$
$$q_{r+1}^{(m)} e_r^{(m)} = q_r^{(m+1)} e_r^{(m+1)}$$

II. Formation of the quantities $\alpha_r^{(m)}$, $\beta_{r-1}^{(m)}$ $m = 0, \cdots, 2n - 2r$ from the relations

$$\alpha_r^{(m)} = e_r^{(m)} + q_{r+1}^{(m)} \qquad r = 0, 1, \cdots, n - 1$$
$$\beta_r^{(m)} = e_{r+1}^{(m)} q_{r+1}^{(m)} \qquad r = 0, 1, \cdots, n - 2$$

III. Generation of the quantities $z^{-1}, z^{-2}, \cdots, z^{-2n}$

IV. Generation of the quantities $c_r z^{-r}$ $r = 0, \cdots, 2n - 1$

V. Generation of the quantities $c_r z^{-r-1}$, $\sum_{s=0}^{m-1} c_s z^{-s-1}$ $r, m = 0, \cdots, 2n - 1$

VI. From the boundary conditions $E_0^{(m)}(z) = \sum_{s=0}^{m-1} c_s z^{-s-1}$ $m = 1, \cdots, 2n$; $E_0^{(0)} = 0$, $E_{1/2}^{(m)}(z) = z^{m+1} c_m^{-1}$ $m = 0, \cdots, 2n - 1$, the formation of the quantities $E_r^{(m)}(z)$ $m = 0, \cdots, 2n - 2r$ and the quantities $E_{r+1/2}^{(m)}(z)$ $m = 0, \cdots, 2n - 2r - 1$, by means of the recursions

$$E_{s+1/2}^{(m)}(z) = E_{s-1/2}^{(m+1)}(z) + \frac{1}{E_s^{(m+1)}(z) - E_s^{(m)}(z)} \qquad s = \tfrac{1}{2}, 1, \cdots, n - \tfrac{1}{2}$$

VII. Generation of the quantities $z - \alpha_r^{(0)}$ $r = 0, \cdots, n - 1$

VIII. Generation of the quantities $z - \alpha_r^{(m)}$

$$m = 0, \cdots, 2n - 2r; r = 0, \cdots, n - 1$$

IX. Generation of the quantities $p_r^{(0)}(z)$ from

$$p_{r+1}^{(0)}(z) = (z - \alpha_r^{(0)}) p_r^{(0)}(z) - \beta_{r-1}^{(0)} p_{r-1}^{(0)}(z)$$
$$r = 1, \cdots, n - 1, \qquad p_0^{(0)}(z) = 1, \qquad p_1^{(0)}(z) = z - q_1^{(0)}.$$

X. Generation of the quantities $o_r^{(0)}(z)$ from

$$o_{r+1}^{(0)}(z) = (z - \alpha_r^{(0)}) o_r^{(0)}(z) - \beta_{r-1}^{(0)} o_{r-1}^{(0)}(z)$$
$$r = 1, \cdots, n - 1, \qquad o_0^{(0)}(z) = 0, \qquad o_1^{(0)} = c_0$$

XI. Generation of the quantities $p_r^{(0)}(z)$ $r = 2, \cdots, n$ from

$$p_r^{(0)}(z) = z p_{r-1}^{(1)}(z) - q_r^{(0)} p_{r-1}^{(0)}(z)$$

XII. Generation of the quantities $p_r^{(m+1)}(z)$ from

$$p_r^{(m+1)}(z) = p_r^{(m)}(z) - e_r^{(m)} p_{r-1}^{(m+1)}(z)$$
$$r = 1, \cdots, n - 1; \qquad m = 0, \cdots, 2n - 2r - 1$$

XIII. Generation of the quantities $r_r^{(0)}(z)$ from

$$r_r^{(0)}(z) = zr_{r-1}^{(1)}(z) - q_r^{(0)}r_{r-1}^{(0)}(z) \qquad r = 2, \cdots, n$$

XIV. Generation of the quantities $r_r^{(m+1)}(z)$ from

$$r_r^{(m+1)}(z) = r_r^{(m)}(z) - e_r^{(m)}r_{r-1}^{(m+1)}(z)$$

$$r = 1, \cdots, n-1; \qquad n = 0, \cdots, 2n - 2r - 1$$

XV. The single division $o_n^{(0)}(z)/p_n^{(0)}(z)$.

XVI. The divisions $o_r^{(0)}(z)/p_r^{(0)}(z) \quad r = 1, \cdots, n$

XVII. The divisions $r_r^{(m)}(z)/p_r^{(m)}(z) \quad r = 1, \cdots, n; m = 0, \cdots, 2n - 2r$

XVIII. Computation of $\dfrac{a_r^{(0)}(z)}{p_r^{(0)}(z)} = \dfrac{a_1}{b_1+} \cdots \dfrac{a_r}{b_r}$, where $a_1 = c_0$; $a_s = -\beta_{s-2}^{(0)}$; $s = 2, \cdots, r$; $b_{s+1} = z - \alpha_s^{(0)}$; $s = 0, \cdots, r-1$, from

$$D_1 = \frac{a_r}{b_r}, \qquad D_{s+1} = \frac{a_{r-s}}{b_{r-s} + D_s}, \qquad D_r = \frac{o_r^{(0)}(z)}{p_r^{(0)}(z)}$$

when $r = n$.

XIX. The use of XVIII. when $r = 1, \cdots, n$

XX. Computation of

$$\frac{r_r^{(m)}(z)}{p_r^{(m)}(z)} = \sum_{s=0}^{m-1} c_s z^{-s-1} + \frac{c_m z^{-m}}{z - \alpha_0^{(m)} -} \frac{\beta_0^{(m)}}{z - \alpha_1^{(m)} -} \cdots \frac{\beta_{r-2}^{(m)}}{z - \alpha_{r-1}^{(m)}}$$

$$= b_0 + \frac{a_1}{b_1+} \cdots \frac{a_r}{b_r}$$

where $b_0 = \sum_{s=0}^{m-1} c_s z^{-s-1}$; $a_1 = c_m z^{-m}$; $a_s = \beta_{s-2}^{(m)}$; $s = 2, \cdots, r$; $b_s = z - \alpha_{s-1}^{(m)}$; $s = 1, \cdots, r$ by

$$D_1 = b_r, \qquad D_{s+1} = b_{r-s} + \frac{a_{r-s+1}}{D_s}, \qquad D_{r+1} = \frac{r_r^{(m)}(z)}{p_r^{(m)}(z)}$$

for $r = 1, \cdots, n-1$; $m = 1, \cdots, 2n - 2r$.

XXI. Computation of the products $c_0\beta_0^{(0)} \cdots \beta_{r-1}^{(0)} \quad r = 1, \cdots, n-1$.

XXII. Computation of the products $z^{-m} c_m\beta_0^{(m)} \cdots \beta_{r-1}^{(m)} \quad r = 1, \cdots, n-1$; $m = 1, \cdots, 2n - 2r$.

XXIII. Computation of the convergents

$$\frac{o_r^{(0)}(z)}{p_r^{(0)}(z)} = \sum_{s=1}^{s=r} \frac{c_0\beta_0^{(0)} \cdots \beta_{s-2}^{(0)}}{p_{s-1}^{(0)}(z)p_s^{(0)}(z)} \qquad r = 1, \cdots, n$$

XXIV. Computation of the convergents

$$\frac{r_r^{(m)}(z)}{p_r^{(m)}(z)} = \sum_{s=0}^{m-1} c_s z^{-s-1} + \sum_{s=1}^{s=r} z^{-m} \frac{c_m \beta_0^{(m)} \cdots \beta_{s-2}^{(m)}}{p_{s-1}^{(m)}(z)p_s^{(m)}(z)} \qquad r = 1, \cdots, n-1 \qquad m = 1, \cdots, 2n - 2r$$

XXV. Computation of $\mathbf{p}_r^{(m)}$ from $\mathbf{p}_0^{(m)} = (1)$; $\mathbf{p}_1^{(m)} = (1, -c_{m+1}/c_m)$; $\mathbf{p}_{r+1}^{(m)} = (\mathbf{p}_r^{(m)}, 0)_{r+2} - \alpha_r^{(m)}(0, \mathbf{p}_r^{(m)})_{r+2} - \beta_{r-1}^{(m)}(0, 0, \mathbf{p}_{r-1}^{(m)})_{r+2}$ $r = 1, \cdots, n-1$

$$\alpha_r^{(m)} = \frac{\mathbf{p}_r^{(m)}\mathbf{c}_r^{(m+r+1)T}}{\mathbf{p}_{r-1}^{(m)}\mathbf{c}_{r-1}^{(m+r+1)T}} - \frac{\mathbf{p}_{r-1}^{(m)}\mathbf{c}_{r-1}^{(m+r+1)T}}{\mathbf{p}_{r-2}^{(m)}\mathbf{c}_{r-2}^{(m+r+1)T}}$$

$$\beta_{r-1}^{(m)} = \frac{\mathbf{p}_r^{(m)}\mathbf{c}_r^{(m+r+1)T}}{\mathbf{p}_{r-1}^{(m)}\mathbf{c}_{r-1}^{(m+r+1)T}} \qquad r = 1, \cdots, n-1$$

for $m = 0$

XXVI. XXV for $m = 0, 2, \cdots, 2n-2; r = 1, 2, \cdots, n-(m/2): m = 1, 3, \cdots, 2n-3; r = 1, 2, \cdots, n-(m+1)/2$

XXVII. Computation of $\mathbf{o}_r^{(0)}$ from $\mathbf{o}_0^{(0)} = 0; \mathbf{o}_1^{(0)} = (c_0)$

$$\mathbf{o}_{r+1}^{(0)} = (\mathbf{o}_r^{(0)}, 0)_{r+1} - \alpha_r^{(0)}(0, \mathbf{o}_r^{(0)})_{r+1} - \beta_{r-1}^{(0)}(0, 0, \mathbf{o}_{r-1}^{(0)})_{r+1}$$

$$r = 1, \cdots, n-1$$

XXVIII. Computation of $\mathbf{p}_r^{(0)}$ $r = 1, \cdots, n$ from $\mathbf{p}_0^{(0)} = (1); \mathbf{p}_1^{(0)} = (1, -c_1/c_0)$

$$\mathbf{p}_{r+1}^{(0)} = (\mathbf{p}_r^{(0)}, 0)_{r+2} - \alpha_r^{(0)}(0, \mathbf{p}_r^{(0)})_{r+2} - \beta_{r-1}^{(0)}(0, 0, \mathbf{p}_{r-1}^{(0)})_{r+2}$$

XXIX. Gauss-Banachiewicz decomposition of the matrix $\mathbf{C}_n^{(0)}$

$$\mathbf{C}_n^{(0)} = \mathbf{H}_n^{(0)}\mathbf{D}_n^{(0)}\mathbf{H}_n^{(0)T}$$

XXX. Decomposition of the matrices $\mathbf{C}_r^{(m)}$ $m = 0, \cdots, 2n-4$; $r = n-2m, n-2m+1$

XXXI. Recovery of the quantities $\beta_{r-1}^{(0)}$ $r = 1, \cdots, n-1$ from $\mathbf{D}_n^{(0)}$ by means of the relations

$$\beta_{r-1}^{(0)} = \frac{c_0\,\beta_0^{(0)} \cdots \beta_{r-1}^{(0)}}{c_0\,\beta_0^{(0)} \cdots \beta_{r-2}^{(0)}}$$

XXXII. Recovery of the quantities $\alpha_r^{(0)}$ $r = 0, \ldots, n-1$ from $\mathbf{H}_n^{(0)}$ by means of the relations

$$\alpha_r^{(0)} = h_{r,r-1}^{(0)} - h_{r+1,r}^{(0)} \qquad \alpha_0^{(0)} = -h_{1,0}^{(0)}$$

XXXIII. Computation of the quantities $\alpha_r^{(m)}$ $r = 0, \cdots, n-1; m = 0, \cdots, 2n-2r$; $\beta_r^{(m)}$ $r = 0, \cdots, n-2; m = 0, \cdots, 2n-2r-1$ from $\mathbf{H}_r^{(m)}$ and $\mathbf{D}_r^{(m)}$ by means of the relationships

$$\alpha_r^{(m)} = h_{r,r-1}^{(m)} - h_{r+1,r}^{(m)} \qquad \beta_{r-1}^{(m)} = \frac{c_m\,\beta_0^{(m)} \cdots \beta_{r-1}^{(m)}}{c_m\,\beta_0^{(m)} \cdots \beta_{r-2}^{(m)}}$$

XXXIV. Computation of the quantities $q_r^{(m)}$ $r = 1, \cdots, n; m = 0, \cdots, 2n-2r$; $e_r^{(m)}$ $r = 1, \cdots, n-1; m = 0, \cdots, 2n-2r-1$ from

$$q_{r+1}^{(m)} = \alpha_r^{(m)} - e_r^{(m)} \qquad e_r^{(m)} = \beta_{r-1}^{(m)}/q_r^{(m)}$$

XXXV. Computation of the matrix $\mathbf{K}_n^{(0)}$ by inversion of $\mathbf{H}_n^{(0)}$

$$\mathbf{H}_n^{(0)}\mathbf{K}_n^{(0)} = \mathbf{I}$$

XXXVI. Computation of the matrices $\mathbf{K}_r^{(m)}$ by inversion of the matrices $\mathbf{H}_r^{(m)}$

$$\mathbf{H}_r^{(m)}\mathbf{K}_r^{(m)} = \mathbf{I} \qquad r = 2, \cdots, n; m = 2n-2r, 2n-2r-1$$

in conjunction with the independent formation of $\mathbf{p}_1^{(2n-2)}$ and $\mathbf{p}_1^{(2n-3)}$

XXXVII. Computation of the quantities $q_r^{(m)}$ $r = 1, \cdots, n; m = 0, \cdots, 2n-2r$ from the relations

$$q_r^{(m)} = k_{r-1,r-2}^{(m)} - k_{r,r-1}^{(m)}$$

XXXVIII. Computation of $\mathbf{p}_r^{(m)}$ $r = 1, \cdots, n; m = 0, \cdots, 2n-2r$ from the recursion

$$\mathbf{p}_r^{(m)} = (\mathbf{p}_{r-1}^{(m+1)}, 0)_{r+1} - q_r^{(m)}(0, \mathbf{p}_{r-1}^{(m)})_{r+1} \qquad \mathbf{p}_1^{(m)} = (1, -q_1^{(m)})$$

XXXIX. Computation of $\mathbf{r}_r^{(m)}$ $r = 1, \cdots, n;\ m = 0, \cdots, 2n - 2r$ from the recursion

$$\mathbf{r}_r^{(m)} = \mathbf{r}_{r-1}^{(m+1)} - q_r^{(m)}(0, \mathbf{r}_{r-1}^{(m)})_{r+m-1}$$

$$\mathbf{r}_0^{(m)} = \mathbf{c}_{m-1}^{(0)} \qquad m = 1, \cdots, 2n - 2r;\ \mathbf{r}_0^{(0)} = 0$$

XL. Computation of $\mathbf{o}_r^{(m)}$ $r = 1, \cdots, n;\ m = 0, \cdots, 2n - 2r$ from the recursion

$$\mathbf{o}_r^{(m)} = c_m \mathbf{p}_{r-1}^{(m+1)} + (0, \mathbf{o}_{r-1}^{(m+1)})_r - q_r^{(m)}(0, \mathbf{o}_{r-1}^{(m)})_r \qquad \mathbf{o}_1^{(m)} = c_m$$

XLI. Computation of $\mathbf{r}_r^{(m)}$ $r = 1, \cdots, n;\ m = 0, \cdots, 2n - 2r$ from

$$\mathbf{r}_r^{(m)} = \mathbf{p}_r^{(m)}[\mathbf{c}_{m-1}^{(0)}]_{r+1,m+r} + (0, \cdots, 0, \mathbf{o}_r^{(m)})_{m+r}$$

XLII. Computation of $\mathbf{o}_n^{(0)}$ from

$$\mathbf{o}_n^{(0)} = \mathbf{p}_n^{(0)}[\mathbf{c}_{n-1}^{(0)}]_{n+1,n}$$

XLIII. Computation of $\mathbf{o}_r^{(0)}$ $r = 1, \cdots, n$ from

$$\mathbf{o}_r^{(0)} = \mathbf{p}_r^{(0)}[\mathbf{c}_{r-1}^{(0)}]_{r+1,r}$$

XLIV. Computation of $\mathbf{r}_r^{(m)}$ $r = 1, \cdots, n-1;\ m = 1, \cdots, 2n - 2r$ from

$$\mathbf{r}_r^{(m)} = \mathbf{p}_r^{(m)}[\mathbf{c}_{r+m-1}^{(0)}]_{r+1,m+r}$$

XLV. Computation of $\mathbf{r}_r^{(0)}$ $r = 1, \cdots, n$ from

$$\mathbf{p}_r^{(0)}[\mathbf{r}_{r-1}^{(0)}]_{r+1,r} = \mathbf{p}_{r-1}^{(0)}[\mathbf{r}_r^{(0)}]_{r,r}$$

XLVI. Determination of $\mathbf{r}_r^{(m)}$ $r = 1, \cdots, n-1;\ m = 1, \cdots, 2n - 2r$ from

$$\mathbf{p}_r^{(m)}[\mathbf{r}_{r-1}^{(m)}]_{r+1,m+r} = \mathbf{p}_{r-1}^{(m)}[\mathbf{r}_r^{(m)}]_{r,m+r}$$

XLVII. The determination of the vectors $\mathbf{d}_{n,n}^{(0)} = \mathbf{o}_n^{(0)}$, $\mathbf{d}_{n,n}^{(0)'} = \mathbf{p}_n^{(0)}$ from the recursions

$$\mathbf{d}_{n,1}^{(0)} = (\beta_{n-2}^{(0)}) \qquad \mathbf{d}_{n,1}^{(0)'} = (1, -\alpha_{n-1}^{(0)}) \qquad \mathbf{d}_{n,s}^{(0)} = \beta_{n-s-1}^{(0)} \mathbf{d}_{n,s-1}^{(0)'},$$

$$\mathbf{d}_{n,s}^{(0)'} = (\mathbf{d}_{n,s-1}^{(0)'}, 0)_{s+1} - \alpha_{n-s}^{(0)}(0, \mathbf{d}_{n,s-1}^{(0)'})_{s+1} + (0, 0, \mathbf{d}_{n,s-1}^{(0)})_{s+1}$$

$$s = 2, \cdots, n;\ \beta_{-1}^{(0)} = c_0$$

XLVIII. The determination of the vectors $\mathbf{d}_{n-r,n}^{(0)} = \mathbf{o}_{n-r}^{(0)}$, $\mathbf{d}_{n-r,n}^{(0)'} = \mathbf{p}_{n-r}^{(0)}$ $r = 0, \cdots, n-1$ from the recursions

$$\mathbf{d}_{r,r}^{(0)} = (\beta_{n-r-2}^{(0)}) \qquad \mathbf{d}_{r,r}^{(0)'} = (1, -\alpha_{n-r-1}^{(0)}) \qquad \mathbf{d}_{r,r+2}^{(0)} = \beta_{n-r-s-2}^{(0)} \mathbf{d}_{r,r+s-1}^{(0)'},$$

$$\mathbf{d}_{r,r+s}^{(0)'} = (\mathbf{d}_{r,r+s-1}^{(0)'}, 0)_{s+2} - \alpha_{n-r-s-1}^{(0)}(0, \mathbf{d}_{r,r+s-1}^{(0)'})_{s+2} + (0, 0, \mathbf{d}_{r,r+s-1}^{(0)})_{s+2}$$

$$r = 0, \cdots, n-1;\ s = 1, \cdots, n-r;\ \beta_{-1}^{(0)} = c_0$$

XLIX. The determination of the vectors $\mathbf{d}_{r,r}^{(m)} = \mathbf{o}_r^{(m)}$, $\mathbf{d}_{r,r}^{(m)'} = \mathbf{p}_r^{(m)}$ from the recursions

$$\mathbf{d}_{r,r}^{(m)} = (\beta_{p-r-2}^{(m)}) \qquad \mathbf{d}_{r,r}^{(m)'} = (1, -\alpha_{p-r-1}^{(m)}) \qquad \mathbf{d}_{r,r+s}^{(m)} = \beta_{p-r-s-2}^{(m)} \mathbf{d}_{r,r+s-1}^{(m)'},$$

$$\mathbf{d}_{r,r+s}^{(m)'} = (\mathbf{d}_{r,r+s-1}^{(m)'}, 0)_{s+2} - \alpha_{p-r-s-1}^{(m)}(0, \mathbf{d}_{r,r+s-1}^{(m)'})_{s+2} + (0, 0, \mathbf{d}_{r,r+s-1}^{(m)})_{s+2}$$

$$\beta_{-1}^{(m)} = c_m$$

$$m = 0, 2, \cdots, 2n-2, \qquad \nu = n - \frac{m}{2} \qquad r = 0, \cdots, \nu - 1,$$

$$m = 1, 3, \cdots, 2n-1, \qquad \nu = n - \frac{(m+1)}{2} \qquad r = 0, \cdots, \nu - 1,$$

$$s = 1, \cdots, \nu - r$$

L. Determination of $\mathbf{p}_{r+1}^{(m)}$ $r = 0, \cdots, n-1; m = 0, \cdots, 2n-2r-2$, from

$$\mathbf{p}_{r-1}^{(m+1)}[\mathbf{p}_{r+1}^{(m)}]_{r,r+2} = (\epsilon_{r,2r}^{(m+1)})^{-1}\{(\mathbf{\varepsilon}_r^{(m+1)}, 0)_{r+3} - (0, \mathbf{\varepsilon}_r^{(m)})_{r+3}\}$$

and of $\mathbf{\varepsilon}_r^{(m)}$ $r = 0, \cdots, n-1; m = 0, \cdots, 2n-2r-1$ from

$$\mathbf{\varepsilon}_r^{(m)} = (0, \mathbf{\varepsilon}_{r-1}^{(m+1)}, 0)_{2r+1} + (k_{r,0}^{(m)}\, i_{r,0}^{(m+1)})^{-1}\mathbf{p}_r^{(m+1)}[\mathbf{p}_r^{(m)}]_{r+1,2r+1}$$

with $\mathbf{p}_1^{(m)} = (1, c_{m+1}/c_m)$, $\mathbf{\varepsilon}_0^{(m)} = (c_{m-1})$.

LI. Construction of the vector $\mathbf{z}_n^{(0)}$

LII. Computation of $p_n^{(0)}(z)$ from $p_n^{(0)}(z) = \mathbf{p}_n^{(0)}\mathbf{z}_n^{(0)T}$

LIII. Computation of $o_n^{(0)}(z)$ from $o_n^{(0)}(z) = \mathbf{o}_n^{(0)}\mathbf{z}_{n-1}^{(0)T}$

LIV. Computation of $p_r^{(0)}(z)$ $r = 1, \cdots, n$ from $p_r^{(0)}(z) = \mathbf{p}_r^{(0)}\mathbf{z}_r^{(0)T}$

LV. Computation of $o_r^{(0)}(z)$ $r = 1, \cdots, n$ from $o_r^{(0)}(z) = \mathbf{o}_r^{(0)}\mathbf{z}_r^{(0)}$

LVI. Construction of the vector $\mathbf{z}_{3n}^{(-n)}$

LVII. Computation of $p_r^{(m)}(z)$ $r = 1, \cdots, n$; $m = 0, \cdots, 2n-2r$ from $p_r^{(m)}(z) = \mathbf{p}_r^{(m)}\mathbf{z}_r^{(0)T}$

LVIII. Computation of $r_r^{(m)}(z)$ $r = 1, \cdots, n$; $m = 0, \cdots, 2n-2r$ from $r_r^{(m)}(z) = \mathbf{r}_r^{(m)}\mathbf{z}_{m+r-1}^{(-m)T}$

The types and numbers of arithmetic operations required in the performance of each of the stages described are given in Table 7.

TABLE 7

	Multiplication	Division	Addition	Subtraction	Add. + Sub.
I.	$n^2 - 2n + 1$	n^2	$n^2 - 3n + 2$	$n^2 - n$	$2n^2 - 4n + 2$
II.	$n - 1$		$n - 1$		$n - 1$
III.	$2n - 1$	1			
IV.	$2n - 1$	1			
V.	$2n$		$2n - 1$		$2n - 1$
VI.		$2n^2 + n$	$2n^2 - n$	$2n^2 - n$	$4n^2 - 2n$
VII.				n	n
VIII.				n^2	n^2
IX.	$2n - 3$			$n - 1$	$n - 1$
X.	$2n - 3$			$n - 2$	$n - 2$
XI.	$2n - 2$			$n - 2$	$n - 2$
XII.	$n^2 - 3n + 2$			$n^2 - n + 1$	$n^2 - n + 1$
XIII.	$2n - 2$			$n - 1$	$n - 1$
XIV.	$n^2 - n$			$n^2 - n$	$n^2 - n$
XV.		1			
XVI.		n			
XVII.		$2n^2 - n$			
XVIII.		$n - 1$	n		n
XIX.		$\frac{1}{2}(n^2 - n)$	$\frac{1}{2}(n^2 + n)$		$\frac{1}{2}(n^2 + n)$
XX.		$\frac{1}{2}(n^2 - n)$	$\frac{1}{2}(n^2 - n)$		$\frac{1}{2}(n^2 - n)$
XXI.	$n - 1$				
XXII.	$n^2 - 3n + 2$				
XXIII.	$n - 1$	n	$n - 1$		$n - 1$

TABLE 7—*Continued*

	Multiplication	Division	Addition	Subtraction	Add. + Sub.
XXIV.	$n^2 - 2n + 1$	n^2	n^2		n^2
XXV.	$2n^2 - 3n + 1$	$2n - 1$	$n^2 - n$	$2n^2 - 3n + 1$	$3n^2 - 4n + 1$
XXVI.	$\frac{1}{3}(4n^3 - 9n^2 + 8n - 3)$	$2n^2 - 2n + 1$	$\frac{1}{3}(2n^3 - 3n^2 + n)$	$\frac{1}{3}(4n^3 - 9n^2 + 8n - 3)$	$2n^3 - 4n^2 + 3n - 1$
XXVII.	$n^3 - 2n + 1$		$n - 1$	$n^2 - 3n + 2$	$n^2 - 2n + 1$
XXVIII.	$n^3 - 2n + 1$	1	$n - 1$	$n^2 - 2n + 1$	$n^2 - n$
XXIX.	$\frac{1}{3}(n^3 + 6n^2 + 5n)$	$\frac{1}{3}(n^2 + n)$		$\frac{1}{3}(n^3 + 3n^2 + 2n)$	$\frac{1}{3}(n^3 + 3n^2 + 2n)$
XXX.	$\frac{1}{12}(n^4 + 8n^3 + 11n^2 + 4n)$	$\frac{1}{6}(2n^3 + 3n^2 + n)$		$\frac{1}{12}(n^4 + 4n^3 + 5n^2 + 2n)$	$\frac{1}{12}(n^4 + 4n^3 + 5n^2 + 2n)$
XXXI.		$n - 1$			
XXXII.				$n - 1$	$n - 1$
XXXIII.		$n^2 - 2n + 1$		$n^2 - 2n + 1$	$n^2 - 2n + 1$
XXXIV.		$n^2 - n$		$n^2 - 2n + 1$	$n^2 - 2n + 1$
XXXV.	$\frac{1}{3}(n^3 - n)$			$\frac{1}{3}(n^3 - n)$	$\frac{1}{3}(n^3 - n)$
XXXVI.	$\frac{1}{12}(n^4 - n^2)$	2		$\frac{1}{12}(n^4 - n^2)$	$\frac{1}{12}(n^4 - n^2)$
XXXVII.				$n^2 - 2n + 1$	$n^2 - 2n + 1$
XXXVIII.	$\frac{1}{3}(2n^3 - 3n^2 + n)$			$\frac{1}{3}(2n^3 - 3n^2 + n)$	$\frac{1}{3}(2n^3 - 3n^2 + n)$
XXXIX.	$\frac{1}{3}(n^3 - n)$			$\frac{1}{3}(n^3 - n)$	$\frac{1}{3}(n^3 - n)$
XL.	$\frac{1}{3}(2n^3 - 3n^2 + n)$		$\frac{1}{3}(2n^3 + 3n^2 + n)$	$\frac{1}{3}(2n^3 - 3n^2 + n)$	$\frac{1}{3}(2n^3 - 3n^2 + n)$
XLI.	$\frac{1}{6}(n^4 + n^2 - n^2 - n)$		$\frac{1}{6}(n^4 + n^3 - n^2 - n$		$\frac{1}{6}(n^4 + n^3 - n^2 - n)$
XLII.	$\frac{1}{3}(n^2 - n)$		$\frac{1}{3}(n^2 - n)$		$\frac{1}{3}(n^2 - n)$
XLIII.	$\frac{1}{3}(n^3 - n)$		$\frac{1}{3}(n^3 - n)$		$\frac{1}{3}(n^3 - n)$
XLIV.	$\frac{1}{6}(n^4 + n^2 - n^2 - n)$		$\frac{1}{6}(n^4 - 3n^3 + 2n^2)$		$\frac{1}{6}(n^4 - 3n^3 + 2n^2)$
XLV.	$\frac{1}{3}(n^3 - n)$		$\frac{1}{3}(n^3 - 7n + 6)$	$\frac{1}{3}(n^3 - n)$	$\frac{1}{3}(n^3 - 4n + 3)$
XLVI.	$\frac{1}{6}(2n^4 + n^3 + n^2 - 4n)$		$\frac{1}{12}(3n^4 + 2n^3 - 9n^2 + 16n - 12)$	$\frac{1}{12}(3n^4 + 2n^3 + 3n^2 - 8n)$	$\frac{1}{6}(3n^4 + 2n^3 - 3n^2 + 4n - 6)$
XLVII.	$n^2 - n$		$\frac{1}{3}(n^2 - 3n + 2)$	$\frac{1}{3}(n^2 + n - 2)$	$n^2 - n$
XLVIII.	$\frac{1}{3}(n^3 - n)$		$\frac{1}{6}(n^3 - 3n^2 + 2n)$	$\frac{1}{6}(n^3 + 3n^2 - 4n)$	$\frac{1}{3}(n^3 - n)$
XLIX.	$\frac{1}{6}(n^4 - n^2)$		$\frac{1}{12}(n^4 - 8n^3 + 23n^2 - 28n + 12)$	$\frac{1}{12}(n^4 + 4n^3 - 7n^2 + 2n)$	$\frac{1}{6}(n^4 - 2n^3 + 8n^2 - 13n + 6)$
L.	$\frac{1}{12}(3n^4 + 8n^3 + 15n^2 - 50n + 24)$	$2n^2 - n$	$\frac{1}{12}(n^4 + 4n^3 - 7n^2 + 2n)$	$\frac{1}{6}(n^4 + 6n^3 - 4n^2 - 9n + 6)$	$\frac{1}{12}(3n^4 + 16n^3 - 15n^2 - 16n + 12)$
LI.	$n - 1$				
LII.	$n - 1$		n		n
LIII.	$n - 1$		$n - 1$		$n - 1$
LIV.	$\frac{1}{3}(n^2 - n)$		$\frac{1}{3}(n^2 + n)$		$\frac{1}{3}(n^2 + n)$
LV.	$\frac{1}{3}(n^2 - n)$		$\frac{1}{3}(n^2 - n)$		$\frac{1}{3}(n^2 - n)$
LVI.	$3n - 2$	1			
LVII.	$\frac{1}{3}(2n^3 - 3n^2 + n)$		$\frac{1}{3}(2n^3 + 3n^2 + n)$		$\frac{1}{3}(2n^3 + 3n^2 + n)$
LVIII.	$\frac{1}{3}(2n^3 - 3n^2 + n)$		$\frac{1}{3}(2n^3 - 3n^2 + n)$		$\frac{1}{3}(2n^3 - 3n^2 + n)$

Since addition and subtraction may be performed with quantities of either sign, the distinction between these two operations is somewhat artificial, and in this instance a column denoting the joint number of additions and subtractions has been added to Table 7. It is assumed that multiplications by zero or unity do not occur, and that divisions in which the quotient or divisor is unity also do not occur. The sign reversal of a number is also not taken into account. It is further assumed that

each of the computations takes place in the most efficient manner possible. For example it is assumed that the quantities

$$\alpha_r^{(m)} = e_r^{(m)} + q_{r+1}^{(m)} \qquad \beta_r^{(m)} = e_{r+1}^{(m)} q_{r+1}^{(m)} \tag{6.16}$$
$$r = 0, \cdots, n-1; m = 1, \cdots, 2n-2r$$

are computed in the process of forming the quantities $e_r^{(m)}$ $r = 1, \cdots, n-1$; $m = 0, \cdots, 2n-2r-1$, $q_r^{(m)}$ $r = 1, \cdots, n$; $m = 0, \cdots, 2n-2r$ by means of the $q-d$ algorithm relationships

$$q_r^{(m)} + e_r^{(m)} = q_r^{(m+1)} + e_{r-1}^{(m+1)} \qquad q_{r-1}^{(m)} e_r^{(m)} = q_r^{(m+1)} e_r^{(m+1)} \tag{6.17}$$

so that the formation of the quantities $\alpha_r^{(m)}$ $r = 0, \cdots, n-1$ $\beta_r^{(m)}$ $r = 0, \cdots, n-2$; $m = 0, \cdots, 2n-2r-4$ from the quantities $e_r^{(m)}$, $q_r^{(m)}$ requires only a further $n-1$ additions and $n-1$ multiplications.

Formulas, the use of which would clearly result in inferior computational practice without the existence of compensatory factors, have not been examined. For example the vectors $\mathbf{r}_r^{(m)}$ $r = 1 \cdots, n$; $m = 0, \cdots, 2n-2r$ should clearly be derived from the recursion

$$\mathbf{r}_r^{(m)} = \mathbf{r}_{r-1}^{(m+1)} - q_r^{(m)}(0, \mathbf{r}_{r-1}^{(m)})_{r+m-1} \tag{6.18}$$

rather than from the recursion

$$\mathbf{r}_{r+1}^{(m)} = (\mathbf{r}_r^{(m)}, 0)_{m+r} - \alpha_r^{(m)}(0, \mathbf{r}_r^{(m)})_{m+r} - \beta_{r-1}^{(m)}(0, 0, \mathbf{r}_{r-1}^{(m)})_{m+r} \tag{6.19}$$

since the first involves only one scalar multiplication and the second two. If the quantities $\alpha_r^{(m)}$, $\beta_{r-1}^{(m)}$ are available the quantities $q_r^{(m)}$ may recursively be determined from them (viz. XXXIV) and if the vectors $\mathbf{p}_r^{(m)}$ are already to hand, the quantities $q_r^{(m)}$ may be derived by (XXXVII).

Feasible methods of obtaining rational approximations, together with the number and type of arithmetic operators involved, now follow (Tables 8–13).

TABLE 8

Single Value $E_n^{(0)}(z)$
A: III, V, VI;
B: I, II, VII, XVIII;
C: I, II, VII, IX, X, XV;
D: I, II, VII, IX, XXI, XXIII;
E: XXIX, XXXV, XLII, LI, LII, LIII, XV:

	Multiplications	Divisions	Additions	Subtractions	Add. + Subs.
A	$4n-1$	$2n^2+n+1$	$2n^2+n-1$	$2n^2-n$	$4n^2-1$
B	n^2-n	n^2-n-1	n^2-n+1	n^2	$2n^2-n+1$
C	n^2+n-3	n^2+1	n^2-2n+1	n^2+2n-3	$2n^2-2$
D	n^2+n-2	n^2+n	n^2-n	n^2+n-1	$2n^2-1$
E	$\frac{1}{6}(2n^3+9n^2+19n-18)$	$\frac{1}{2}(n^2+n+2)$	$\frac{1}{6}(n^3+6n^2-n)$	$\frac{1}{6}(2n^3+3n^2+n)$	$\frac{1}{3}(n^3+3n^2+5n-3)$

TABLE 9

Diagonal of Values $E_r^{(0)}(z)$ $r = 1, \cdots, n$
A:
F: I, II, VII, XIX;
G: I, II, VII, IX, X, XVI;
D:
H: XXIX, XXXV, XLIII, LI, LIV, LV, XVI:

	Multiplications	Divisions	Additions	Subtractions	Adds. + Subs.
A	$4n - 1$	$2n^2 + n + 1$	$2n^2 + n - 1$	$2n^2 - n$	$4n^2 - 1$
F	$n^2 - n$	$\frac{1}{2}(3n^2 - 1)$	$\frac{1}{2}(3n^2 - 3n + 2)$	n^2	$\frac{1}{2}(5n^2 - 3n + 2)$
G	$n^2 + 3n - 6$	$n^2 + n$	$n^2 - 2n + 1$	$n^2 + 2n - 3$	$2n^2 - 2$
D	$n^2 + 3n - 5$	$n^2 + n$	$n^2 - n$	$n^2 + n - 1$	$2n^2 - 1$
H	$\frac{1}{2}(n^2 + 4n^2 + n - 2)$	$\frac{1}{2}(n^2 + 3n)$	$\frac{1}{2}(n^2 + 6n^2 - n)$	$\frac{1}{2}(n^2 + 3n^2 + n)$	$\frac{1}{2}(n^2 + 3n^2)$

TABLE 10

Array of Values $E_r^{(m)}(z)$ $r = 0, \cdots, n; m = 0, \cdots, 2n - 2r$
A:
I: I, II, IV, V, VIII, XX;
J: I, III, V, VIII, XI, XII, XXI, XXII, XXIII, XXIV;
K: I, III, V, XI, XII, XIII, XIV, XVII;
L: XXX, XXXVI, XLIV, LVI, LVII, LVIII, XVII;

	Multiplications	Divisions	Additions	Subtractions	Adds. + Subs.
A	$4n - 1$	$2n^2 + n + 1$	$2n^2 + n - 1$	$2n^2 - n$	$4n^2 - 1$
I	$n^2 + 2n - 1$	$\frac{1}{2}(n^2 + 3n^2 - n + 3)$	$\frac{1}{2}(n^2 + 3n^2 - n)$	$2n^2 - n$	$\frac{1}{2}(n^2 + 9n^2 - 4n)$
J	$3n^2 - 2n + 1$	$2n^2 + n + 1$	$2n^2$	$3n^2 - n - 1$	$5n^2 - n - 1$
K	$3n^2 + 2n - 2$	$3n^2 - n + 1$	$n^2 - n + 1$	$3n^2 - n - 2$	$4n^2 - 2n - 1$
L	$\frac{1}{12}(4n^4 + 20n^2 + 5n^2 + 67n - 24)$	$\frac{1}{6}(2n^2 + 21n^2 - 5n + 24)$	$\frac{1}{12}(2n^4 + 4n^2 + 13n^2 - 7n + 24)$	$\frac{1}{6}(n^4 + 2n^2 + 8n^2 - 5n)$	$\frac{1}{6}(4n^4 + 8n^2 + 29n^2 - 17n + 12)$

TABLE 11

Single Rational Function $\mathfrak{o}_n^{(0)}$: $\mathfrak{p}_n^{(0)}$
A′: XXV, XXVII;
B′: XXV, XLII;
C′: I, II, XXVII, XXVIII;
D′: XXIX, XXXI, XXXII, XXVII, XXVIII;
E′: XXIX, XXXI, XXXII, XXXV, XXVII;
F′: XXIX, XXXV, XLII;
G′: I, II, XXVIII, XLII;
H′: I, II, XXVIII, XLV;
I′: I, II, XLVII;

	Multiplications	Divisions	Additions	Subtractions	Adds. + Subs.
A′	$3n^2 - 5n + 2$	$2n - 1$	$n^2 - 1$	$3n^2 - 6n + 3$	$4n^2 - 6n + 2$
B′	$\frac{1}{2}(5n^2 - 7n + 2)$	$2n - 1$	$\frac{1}{2}(n^2 - n)$	$2n^2 - 3n + 1$	$\frac{1}{2}(7n^2 - 9n + 2)$
C′	$3n^2 - 5n + 2$	$n^2 + 1$	$n^2 - 1$	$3n^2 - 6n + 3$	$4n^2 - 6n + 2$
D′	$\frac{1}{6}(n^2 + 18n^2 - 19n + 12)$	$\frac{1}{2}(n^2 + 3n)$	$2n - 2$	$\frac{1}{6}(n^2 + 15n^2 - 22n + 12)$	$\frac{1}{6}(n^2 + 15n^2 - 10n)$
E′	$\frac{1}{6}(n^2 + 6n^2 - 4n + 3)$	$\frac{1}{2}(n^2 + 3n - 2)$	$n - 1$	$\frac{1}{6}(2n^2 + 9n^2 - 11n + 6)$	$\frac{1}{6}(2n^2 + 9n^2 - 5n)$
F′	$\frac{1}{4}(2n^2 + 9n^2 + n)$	$\frac{1}{2}(n^2 + n)$	$\frac{1}{4}(n^2 - n)$	$\frac{1}{4}(2n^2 + 3n^2 + n)$	$\frac{1}{4}(n^2 + 3n^2 - n)$
G′	$\frac{1}{2}(5n^2 - 7n + 2)$	$n^2 + 1$	$\frac{1}{2}(n^2 - n)$	$2n^2 - 3n + 1$	$\frac{1}{2}(7n^2 - 9n + 2)$
H′	$\frac{1}{6}(n^2 + 6n^2 - 10n + 3)$	$n^2 + 1$	$\frac{1}{6}(n^2 + 6n^2 - 13n + 6)$	$\frac{1}{6}(n^2 + 12n^2 - 19n + 6)$	$\frac{1}{6}(n^2 + 9n^2 - 16n + 6)$
I′	$2n^2 - 2n$	n^2	$\frac{1}{2}(3n^2 - 7n + 4)$	$\frac{1}{2}(3n^2 - n - 2)$	$3n^2 - 4n + 1$

TABLE 12

Diagonal of Rational Functions $o_r^{(0)} : p_r^{(0)}$ $r = 1, \cdots, n$

A′:
J′: XXV, XLIII;
D′:
E′:
K′: XXIX, XXXV, XLIII;
L′: I, II, XXVIII, XLIII;
H′:
M′: I, II, XLVIII;

	Multiplications	Divisions	Additions	Subtractions	Adds. + Subs.
A′	$3n^2 - 5n + 2$	$2n - 1$	$n^2 - 1$	$3n^2 - 6n + 3$	$4n^2 - 6n + 2$
J′	$\frac{1}{6}(n^3 + 12n^2 - 19n + 6)$	$2n - 1$	$\frac{1}{6}(n^3 + 6n^2 - 7n)$	$2n^2 - 3n + 1$	$\frac{1}{6}(n^3 + 18n^2 - 25n + 6)$
D′	$\frac{1}{6}(n^3 + 18n^2 - 19n + 12)$	$\frac{1}{2}(n^2 + 3n)$	$2n - 2$	$\frac{1}{6}(n^3 + 15n^2 - 22n + 12)$	$\frac{1}{6}(n^3 + 15n^2 - 10n)$
E′	$\frac{1}{6}(n^3 + 6n^2 - 4n + 3)$	$\frac{1}{2}(n^2 + 3n - 2)$	$n - 1$	$\frac{1}{6}(2n^3 + 9n^2 - 11n + 6)$	$\frac{1}{6}(2n^3 + 9n^2 - 5n)$
K′	$\frac{1}{6}(n^3 + 2n^2 + n)$	$\frac{1}{2}(n^2 + n)$	$\frac{1}{6}(n^3 - n)$	$\frac{1}{6}(2n^3 + 3n^2 + n)$	$\frac{1}{6}(n^3 + n^2)$
L′	$\frac{1}{6}(n^3 + 12n^2 - 19n + 6)$	$n^2 + 1$	$\frac{1}{6}(n^3 + 6n^2 - 13n + 6)$	$2n^2 - 3n + 1$	$\frac{1}{6}(n^3 + 18n^2 - 31n + 12)$
H′	$\frac{1}{3}(n^3 + 6n^2 - 10n + 3)$	$n^2 + 1$	$\frac{1}{6}(n^3 + 6n^2 - 13n + 6)$	$\frac{1}{6}(n^3 + 12n^2 - 19n + 6)$	$\frac{1}{3}(n^3 + 9n^2 - 16n + 6)$
M′	$\frac{1}{3}(n^3 + 3n^2 - 4n)$	n^2	$\frac{1}{6}(n^3 + 3n^2 - 16n + 12)$	$\frac{1}{6}(n^3 + 9n^2 - 10n)$	$\frac{1}{3}(n^3 + 6n^2 - 13n + 6)$

TABLE 13

Complete Array of Rational Functions $r_r^{(m)} : p_r^{(m)}$ $r = 0, \cdots, n$; $m = 0, \cdots, 2n - 2r$

N′: XXVI, XXXVII, XXXIX;
O′: XXVI, XXXVII, XL, XLI;
P′: I, XXXVIII, XXXIX;
Q′: I, XXXVIII, XL, XLI;
R′: XXX, XXXIII, XXXIV, XXXVIII, XXXIX;
S′: XXX, XXXIII, XXXIV, XXXVIII, XL, XLI;
T′: XXX, XXXVI, XXXVII, XXXIX;
U′: XXX, XXXVI, XXXVII, XL, XLI;
V′: XXX, XXXVI, XLIV;
W′: XXX, XXXVI, XLVI;
X′: I, II, XLIX;
Y′: XXXVII, L, XXXIX;
Z′: XXXVII, L, XL, XLI;

	Multiplications	Divisions	Additions	Subtractions	Adds. + Subs.
N′	$\frac{1}{3}(11n^3 - 18n^2 + 13n - 6)$	$2n^2 - 2n + 1$	$\frac{1}{3}(2n^3 - 3n^2 + n)$	$\frac{1}{6}(11n^3 - 12n^2 + n)$	$\frac{1}{2}(5n^3 - 6n^2 + n)$
O′	$\frac{1}{6}(n^4 + 9n^3 - 19n^2 + 15n - 6)$	$2n^3 - 2n + 1$	$\frac{1}{4}(n^4 + 7n^3 - 10n^2 + 2n)$	$\frac{1}{6}(10n^3 - 15n^2 + 5n)$	$\frac{1}{6}(n^4 + 17n^3 - 25n^2 + 7n)$
P′	$\frac{1}{6}(5n^3 + 3n^2 - 14n + 6)$	n^2	$n^2 - 3n + 2$	$\frac{1}{6}(5n^3 + 3n^2 - 8n)$	$\frac{1}{6}(5n^3 + 9n^2 - 26n + 12)$
Q′	$\frac{1}{6}(n^4 + 7n^3 - 4n^2 - 10n + 6)$	n^2	$\frac{1}{4}(n^4 + 3n^3 + 2n^2 - 18n + 12)$	$\frac{2}{3}(n^3 - n)$	$\frac{1}{4}(n^4 + 7n^3 + 2n^2 - 22n + 12)$
R′	$\frac{1}{12}(n^4 + 18n^3 + 5n^2)$	$\frac{1}{6}(2n^3 + 15n^2 - 17n + 6)$		$\frac{1}{12}(n^4 + 14n^3 + 23n^2 - 50n + 24)$	$\frac{1}{12}(n^4 + 14n^3 + 23n^2 - 50n + 24)$
S′	$\frac{1}{12}(3n^4 + 22n^3 - 9n^2 + 8n)$	$\frac{1}{6}(2n^3 + 15n^2 - 17n + 12)$	$\frac{1}{6}(n^4 + 3n^3 - 4n^2)$	$\frac{1}{12}(n^4 + 12n^3 + 5n^2 - 18n + 12)$	$\frac{1}{12}(3n^4 + 18n^3 - 3n^2 - 18n + 12)$
T′	$\frac{1}{6}(n^4 + 7n^3 + 5n^2 - n)$	$\frac{1}{2}(2n^3 + 3n^2 + n + 12)$		$\frac{1}{6}(n^4 + 5n^3 + 8n^2 - 14n + 6)$	$\frac{1}{6}(n^4 + 5n^3 + 8n^2 - 14n + 6)$

TABLE 13—*Continued*

	Multiplications	Divisions	Additions	Subtractions	Adds. + Subs.
U′	$\frac{1}{6}(2n^4 + 9n^3 - 2n^2 + 3n)$	$\frac{1}{6}(2n^3 + 3n^2 + n + 12)$	$\frac{1}{6}(n^4 + 3n^3 - 4n^2)$	$\frac{1}{6}(n^4 + 7n^3 + 5n^2 - 13n + 6)$	$\frac{1}{6}(2n^4 + 10n^3 + n^2 - 13n + 6)$
V′	$\frac{1}{6}(2n^4 + 5n^3 + 4n^2 + n)$	$\frac{1}{6}(2n^3 + 3n^2 + n + 12)$	$\frac{1}{6}(n^4 - 3n^3 + 2n^2)$	$\frac{1}{6}(n^4 + 2n^3 + 2n^2 + n)$	$\frac{1}{6}(2n^4 - n^3 + 4n^2 + n)$
W′	$\frac{1}{6}(3n^4 + 5n^3 + 6n^2 - 2n)$	$\frac{1}{6}(2n^3 + 3n^2 + n + 12)$	$\frac{1}{12}(3n^4 + 2n^3 - 9n^2 + 16n - 12)$	$\frac{1}{12}(5n^4 + 6n^3 + 7n^2 - 6n)$	$\frac{1}{12}(5n^4 + 6n^3 + n^2 + 6n - 6)$
X′	$\frac{1}{6}(n^4 + 5n^2 - 6n)$	n^2	$\frac{1}{12}(n^4 - 8n^3 + 35n^2 - 64n + 36)$	$\frac{1}{12}(n^4 + 4n^3 + 5n^2 - 10n)$	$\frac{1}{6}(n^4 - 2n^3 + 20n^2 - 37n + 18)$
Y′	$\frac{1}{12}(3n^4 + 14n^3 + 15n^2 - 56n + 24)$	$2n^2 - n$	$\frac{1}{12}(n^4 + 4n^3 - 7n^2 + 2n)$	$\frac{1}{6}(n^4 + 9n^3 + 8n^2 - 24n + 12)$	$\frac{1}{12}(3n^4 + 22n^3 - 9n^2 - 46n + 24)$
Z′	$\frac{1}{12}(5n^4 + 18n^3 + n^2 - 48n + 24)$	$2n^2 - n$	$\frac{1}{12}(3n^4 + 10n^3 - 15n^2 + 2n)$	$\frac{1}{6}(n^4 + 8n^3 - n^2 - 20n + 12)$	$\frac{1}{12}(5n^4 + 26n^3 - 17n^2 - 38n + 24)$

It appears from Tables 8 to 10 that, in terms of computational effort alone, a single entry in the E-array for a fixed value of the argument is most economically computed by means of the non-linear recursion

$$D_1 = \frac{\beta_{n-2}^{(0)}}{z - \alpha_{n-1}^{(0)}} \qquad D_{s-1} = \frac{a_{n-s}}{b_{n-s} + D_s}$$

$$a_n = c_0;\ a_s = \beta_{s-2}^{(0)};\ s = 2, \cdots, n;\ b_s = z - \alpha_{s-1}^{(0)};\ s = 1, \cdots, n$$

in conjunction with the q-d algorithm

$$q_r^{(m)} + e_r^{(m)} = q_r^{(m+1)} + e_{r+1}^{(m+1)} \qquad q_{r+1}^{(m)} e_r^{(m)} = q_r^{(m+1)} e_r^{(m+1)} \tag{6.21}$$

and the numerical formation of the quantities

$$\alpha_r^{(0)} = e_r^{(0)} + q_{r+1}^{(0)} \qquad \beta_{r-1}^{(0)} = e_r^{(0)} q_r^{(0)} \qquad r = 1, \cdots, n-1; \tag{6.22}$$

that a diagonal of quantities $E_r^{(0)}(z)$ $r = 1, \cdots, n$ is best computed by use of the q-d algorithm, the formation of the quantities (6.22), the twin recursions

$$p_{r+1}^{(0)}(z) = (z - \alpha_r^{(0)}) p_r^{(0)}(z) - \beta_{r-1}^{(0)} p_{r-1}^{(0)}(z), \qquad r = 1, \cdots, n$$
$$p_0^{(0)}(z) = 1, \qquad p_1^{(0)}(z) = z - q_1^{(0)} \tag{6.23}$$

$$o_{r+1}^{(0)}(z) = (z - \alpha_r^{(0)}) o_r^{(0)}(z) - \beta_{r-1}^{(0)} o_{r-1}^{(0)}(z), \qquad r = 1, \cdots, n$$
$$o_0^{(0)}(z) = 0, \qquad o_1^{(0)}(z) = c_0 \tag{6.24}$$

and the n divisions

$$E_r^{(0)}(z) = \frac{o_r^{(0)}(z)}{p_r^{(0)}(z)} \qquad r = 1, \cdots, n, \tag{6.25}$$

while the complete E-array of numerical values is best computed by evaluating the partial sums $E_0^{(m)}(z) = \sum_{s=0}^{m-1} c_s z^{-s-1}$ $m = 1, \cdots, 2n$, the quantities $E_{1/2}^{(m)}(z) = z^m c_m^{-1}$ and applying the ϵ-algorithm

$$E_{r+1/2}^{(m)}(z) = E_{r-1/2}^{(m+1)}(z) = \frac{1}{E_r^{(m+1)}(z) - E_r^{(m)}(z)} \qquad r = \tfrac{1}{2}, 1, \tfrac{3}{2}, \cdots \tag{6.26}$$

Tables 11 to 13 show, again in terms of computational effort alone, that the single rational expression $E_n^{(0)}(z)$ be best constructed by determining the quantities $\alpha_r^{(0)}$, $\beta_{r-1}^{(0)}$ by means of the relationships

$$I\{t^r p_r^{(0)}(t)\} = c_0\beta_0^{(0)} \cdots \beta_{r-1}^{(0)}$$
$$I\{t^{r+1}p_0^{(0)}(t)\} = c_0\beta_0^{(0)} \cdots \beta_{r-1}^{(0)}\{\alpha_0^{(0)} + \cdots + \alpha_r^{(0)}\} \tag{6.27}$$

in conjunction with the recursion (6.23), and of the vector $\mathbf{o}_n^{(0)}$ from

$$\mathbf{o}_n^{(0)} = \mathbf{p}_n^{(0)}[\mathbf{c}_{n-1}^{(0)}]_{n+1,n} \tag{6.28}$$

that the sequence of rational expressions $E_r^{(0)}(z) \quad r = 1, \cdots, n$ is best constructed by use of relationships (6.27), in conjunction with the recursion (6.23), determining the polynomials $o_r^{(0)}(z)$ by means of the second recursion (6.24); while the complete E-array of rational expressions is most efficiently constructed by use of the q-d algorithm (6.22), in conjunction with the relationships

$$p_r^{(m)}(z) = zp_{r-1}^{(m+1)}(z) - q_r^{(m)}p_{r-1}^{(m)}(z) \qquad p_1^{(m)}(z) = z - q_1^{(m)} \tag{6.29}$$

$$r_r^{(m)}(z) = zr_{r-1}^{(m+1)}(z) - q_r^{(m)}r_{r-1}^{(m)}(z) \qquad r_0^{(m)}(z) = \sum_{s=0}^{m-1} c_s z^{-s-1}, \qquad r_0^{(0)}(z) = 0 \tag{6.30}$$

Many electronic digital computers are equipped with highly flexible interpretive codes which are designed for carrying out standard matrix operations upon arrays of numbers. Many of the methods described may be susceptible to the use of such a code, and this consideration may cause the preceding judgements to be modified. The most suitable method is then largely determined by the exigencies of the particular code used. (This explains the presence, in the inquiry, of such methods as E, H, and L which would otherwise be rejected outright.) The time taken for the operation of such a program may be estimated from the data given in the table, and a knowledge of the threshold time of each instruction.

Table 7 does not contain data relating to the direct solution of the equations

$$\begin{aligned}
&k_{n,n}^{(m)}c_0 = i_{n,m+n-1}^{(m)}\\
&k_{n,n-1}^{(m)}c_0 + k_{n,n}^{(m)}c_1 = i_{n,m+n-2}^{(m)}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&k_{n,0}^{(m)}c_0 + k_{n,1}^{(m)}c_1 + \cdots + k_{n,n}^{(m)}c_n = i_{n,m-1}^{(m)}\\
&k_{n,0}^{(m)}c_1 + k_{n,1}^{(m)}c_2 + \cdots + k_{n,n}^{(m)}c_{n+1} = i_{n,m-2}^{(m)}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&k_{n,0}^{(m)}c_{m-1} + k_{n,1}^{(m)}c_m + \cdots + k_{n,n}^{(m)}c_{n+m-1} = i_{n,0}^{(m)}\\
&k_{n,0}^{(m)}c_m + k_{n,1}^{(m)}c_{m+1} + \cdots + k_{n,n}^{(m)}c_{n+m} = 0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&k_{n,0}^{(m)}c_{n+m-1} + k_{n,1}^{(m)}c_{n+m} + \cdots + k_{n,n}^{(m)}c_{2n+m-1} = 0\\
&k_{n,n}^{(m)} = 1;
\end{aligned} \tag{6.31}$$

the details, of course, depend upon the method chosen for the solution of the equations. The solution of the set (6.31) is equivalent to the Gauss-Banachiewicz decomposition of the matrix

$$\mathbf{C}_n^{(m)} = \mathbf{H}_n^{(m)}\mathbf{D}_n^{(m)}\mathbf{H}_n^{(m)T} \tag{6.32}$$

the inversion of the matrix $\mathbf{H}_n^{(m)}$ to give $\mathbf{K}_n^{(m)}$

$$\mathbf{H}_n^{(m)}\mathbf{K}_n^{(m)} = I \tag{6.33}$$

and the determination of the vector $\mathbf{r}_n^{(m)}$ from

$$\mathbf{r}_n^{(m)} = p_n^{(m)}[\mathbf{c}_{m+n-1}^{(0)}]_{n+1,m+n}\,. \tag{6.34}$$

This method (method V' above), if the steps involved are already incorporated in the interpretive code, is probably the one requiring the fewest instructions, and doubtless that consuming the most machine time.

(It may at first seem a little curious that the set of equations (6.31) does not involve the quantity c_{m+2n}, whereas the matrix $\mathbf{C}_n^{(m)}$ does. The quantity c_{m+2n}, however, only enters into the determination of $\mathbf{H}_n^{(m)}$ as a scaling factor of the last row, and this is removed by imposing the condition that the diagonal elements of $\mathbf{H}_n^{(m)}$ are all unity. The quantity c_{m+2n} does not therefore enter into the determination of the matrix $\mathbf{K}_n^{(m)}$.)

Apart from the considerations mentioned above, only those methods are described which might genuinely be suggested as the most economical. Many of the conclusions which may be deduced from a survey of Tables 8 to 13 might also have been arrived at after a preliminary discussion of the steps listed as I) to LVIII) but this discussion would have been so beset by qualification and modification that the most effective method of comparison was thought to be the straight forward enumeration of all feasible possibilities.

7. Checking and Correction of the Approximations. Since all the methods described in the last section are capable of producing the same final result, the computations produced by the use of one may be checked by application of any of the others, supplemented if necessary, by use of any of the theoretical results given in the text. In particular, attention is drawn to the strength of the use of the system of equations (6.31) as a final check.

Many of the methods entailed the computation of a sequence of polynomials $p_r^{(m)}(z)$ $r = 0, 1, \cdots$. The computation of the coefficients of these polynomials may be checked and corrected by a direct appeal to the orthogonality properties. Assuming the vectors $\mathbf{p}_s^{(m)}$ $s = 0, 1, \cdots, r-1$, to have been correctly computed, a corrected version $\mathbf{p}_{r_{corr.}}^{(m)}$ of the vector $\mathbf{p}_r^{(m)}$ may be produced from the formula

$$\mathbf{p}_{r_{corr}}^{(m)} = \mathbf{p}_r^{(m)} - \sum_{s=0}^{r-1} \gamma_{r,s}\mathbf{p}_s^{(m)} \tag{7.1}$$

where

$$\gamma_{r,s} = \mathbf{p}_r^{(m)}\mathbf{c}_r^{(m)}(\mathbf{p}_s^{(m)}, 0, \cdots, 0)_{r+1} \tag{7.2}$$

If the computation of the vector $\mathbf{p}_r^{(m)}$ has been performed without error, then it will transpire that

$$\gamma_{r,s} = 0 \qquad s = 0, 1, \cdots, r-1. \tag{7.3}$$

The corrected vector $\mathbf{p}_r^{(m)}$ may now be used to check to computation of the vector $\mathbf{p}_{r+1}^{(m)}$ and so on.

Readers familiar with the computational finesses of Lanczos' [22] biortho-

gonalization algorithm for determining the characteristic polynomial of a matrix will readily perceive that the use of equations (7.1) and (7.2) is a straightforward adaptation of a similar technique which relates to the use of this algorithm. Details of connections between this algorithm, the *q-d* algorithm and certain others, which are significant in a theoretical context, are to be found in [10].

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On the Utility of Newton's Method for Computing Complex Roots of Equations

By I. M. Longman

1. Introduction. With the advent of modern high speed computers, methods of computation which were once regarded as being too laborious to be of practical use can very profitably be resuscitated. This applies particularly to iterative methods which are very simple to program for a machine, but which might be very tedious for a human operator to carry out.

For example it has been noted by Whittaker and Robinson [1] that Newton's method of numerical solution of equations is theoretically applicable to complex as well as real roots, but would be extremely laborious to apply.

Despite the simplicity of the method it does not appear to have had the application it deserves, other authors preferring more complicated methods which involve less cumbersome numerical work. An interesting method is described by Ward [2].

2. Examples. Consider the equation

$$y = f(x) \equiv x^2 + 1 = 0. \tag{1}$$

The roots of this equation are known, of course, to be $x = \pm i$, but it is instructive to apply Newton's method and obtain successive approximations to the solution.

We have

$$f'(x) = 2x, \tag{2}$$

and so our recursion formula is

$$x_{n+1} = (x_n - 1/x_n)/2. \tag{3}$$

Now it is clear that the real axis separates those points of the complex plane nearer to one root from those nearer to the other. Also if we start from a first approximation x_0 which is real, all the successive x_n's must be real, in accordance with equation (3). Thus starting with a real x_0 does not lead to a convergence of the iteration. However if we start with any x_0 in the upper half-plane, the sequence (3) converges to the root $x = +i$ and if we start with any x_0 in the lower half-plane the sequence converges to the root $x = -i$. That this is true may be seen from the following:

If we start with $x_0 = a + ib$,

$$x_1 = (a/2)[1 - 1/(a^2 + b^2)] + i(b/2)[1 + 1/(a^2 + b^2)], \tag{4}$$

so that the imaginary part remains positive if it is initially so, and likewise remains negative if it is initially so.

For example, starting with $x_0 = 1 + i$, we have the sequence

$$x_1 = 0.25 + 0.75i$$

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$$x_2 = -0.075 + 0.975i$$

$$x_3 = +0.001715686 + 0.99730392i \text{ etc.},$$

which is clearly converging rapidly to the root $+i$.

The second example is an equation which arose in the course of the author's work on the reflection of wave-pulses in elastic solids (to be published elsewhere), and while it was solved easily on an electronic computer (WEIZAC)* by Newton's method, it would seem difficult to solve it by other means. The equation is

$$f(x) \equiv (x^2 + a^2)^{1/2} + \alpha(x^2 + b^2)^{1/2} - i\rho x = \tau, \tag{5}$$

where $b > a > 0$, $\alpha > 0$, $\rho > 0$, $\tau > 0$, and the x-plane is supposed cut along the imaginary axis between $\pm ib$ so as to make $f(x)$ single valued, and so that the radicals reduce to positive arithmetic square roots on the positive real axis. Then it can be shown [3] that if $\tau > \tau_0$ where

$$\tau_0 = (a^2 - v_0^2)^{1/2} + \alpha(b^2 - v_0^2)^{1/2} + \rho v_0 \tag{6}$$

and v_0 is the (unique) root of

$$v(a^2 - v^2)^{-1/2} + \alpha v(b^2 - v^2)^{-1/2} = \rho \qquad (0 < v_0 < a) \tag{7}$$

then (5) has a unique (complex) root $x = X$. Furthermore X lies in the positive quadrant of the x-plane.

The problem was to compute X on an electronic computer for given values of the (real) parameters a, α, b, ρ, τ. By applying Newton's method a recursion formula was found in the usual way. This method was applied to the equation

$$(x^2 + 1)^{1/2} + (x^2 + \tfrac{1}{3})^{1/2} - ix = 2,$$

and starting from $x_0 = 1 + i$ the following iterates were obtained.

$$x_1 = .49278\ 43132 + .44404\ 59775i$$

$$x_2 = .37934\ 25961 + .38501\ 14705i$$

$$x_3 = .36899\ 47789 + .38109\ 03135i$$

$$x_4 = .36889\ 46156 + .38106\ 80636i$$

$$x_5 = .36889\ 46067 + .38106\ 80642i$$

$$x_6 = .36889\ 46067 + .38106\ 80642i$$

It is seen that the iteration converges extremely rapidly, and with many different values of the parameters the same rapid convergence was found.

For a complete discussion of the conditions for convergence of Newton's method, and uniqueness of the root, reference may be made to Householder [4].

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* WEIZAC is the name of the electronic computer at the Weizmann Institute of Science.

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A Problem in Abelian Groups, with Application to the Transposition of a Matrix on an Electronic Computer

By Gordon Pall and Esther Seiden

1. Introduction. Mr. G. A. Westlund of "Mura" (Midwestern Universities Research Association) was asked to formulate a code to transpose a matrix stored in the memory of IBM 704, using very little additional space. It appeared that such a code, by Mr. William Shooman of General Electric in Evendale, was already available and had been distributed on June 15, 1957, to SHARE members of IBM 704. Mr. Westlund asked whether there exists a more efficient method for this purpose.

A matrix of m rows and n columns is stored in the computer with the elements of each column listed in order and followed by those of the next column. The positions can be numbered as $mj + i$ $(i = 0, 1, \cdots, m - 1; j = 0, 1, \cdots, n - 1)$. To transpose the matrix, the element in position $mj + i$ must be moved to position $ni + j$. The key remark is that the new position number is obtained from the old by multiplication by n, and reducing mod $N(= mn - 1)$. Starting with any element (which we will call a *leader*), we multiply its position number by n and replace mod N, repeat this operation with the new position number again and again, and thus have a cycle of elements which are permuted cyclically in the process of transposing a matrix. If v is the g.c.d. of any of the position numbers and N, then the number of elements in the cycle is equal to the least positive integer r for which $n^r \equiv 1 (\mathrm{mod}\ N/v)$.

The questions then arise: (a) can a method be devised of choosing one and only one leader in every cycle; and (b) if this is done, will the new method of transposing a matrix compare favorably in machine time with the existing method? Both questions are answered affirmatively in this note.

A program for transposing a matrix, once a set of leaders is given, was constructed at our request by G. A. Westlund, under the direction of M. R. Storm, head of the computing section at Mura. The suggestion was made that our method for forming a set of leaders (see §2) might also be programmed by building a table of indices and primitive roots into the computer. We felt that it would be more economical to carry out the construction of a set of leaders by hand. The construction of a table of leaders for all pairs m, n with certain properties is under investigation.

In transposing a matrix by the Mura program, the machine time is much smaller

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than that by the General Electric method (Matrix Transposed on Itself, Shooman at Evendale, June 21, 1957). By the latter method the total number of switches is $m(m-1)n(n-1)/4$, and is roughly proportional to the square of the number of elements of the matrix. By our method the number of switches is roughly proportional to the number of elements of the matrix. Here are some examples of machine times.

Size of matrix.	Time by Mura Program.	Time by G. E. program.
8 by 7	Too small to measure	Too small to measure
21 by 14	Too small to measure	2.5 seconds
18 by 15	Too small to measure	2 seconds
31 by 8	Too small to measure	2 seconds
24 by 35	Less than a second	23 seconds
23 by 25	Less than a second	20.5 seconds
25 by 31	Less than a second	21 seconds
75 by 73	Less than five seconds	17.5 minutes.

2. Background Theory. Let G be the group of residues prime to N modulo N, and let n be an integer prime to N. We will give an algorithm for choosing a set $S(n, N)$, containing exactly one element of each coset in G of the group of powers of n modulo N.

It seems necessary to remark that it seems not to be trivial—as we at first supposed—to devise a *general* method of choosing a complete set of representatives of a group modulo a subgroup—if one does not wish to take the time to write out coset after coset, and *see* what remains. Notice for example that if p and q are distinct primes, and if p divides $q-1$, the solvability for x of $an^x \equiv b \pmod{p^r}$ and for y of $an^y \equiv b \pmod{q^s}$ may impose on x and y inconsistent conditions modulo p,—and so a and b can be in the same coset mod p^r and mod q^s without being so for their product. It will be seen in the vector representation which follows that now one and now another of the components dominates the congruence conditions, and an unsystematic analysis may be difficult. Perhaps indeed our algorithm may have value beyond this present application.

As is well known, if $p = p_i$ is an odd prime, and $t = t_i$ a positive integer, there is an integer $g = g_i$ (a primitive root of p^t) such that every residue k prime to p modulo p^t is associated uniquely with an index e mod N_i, where

$$N_i = p_i^{t_i-1}(p_i - 1), \tag{1}$$

such that $k \equiv g^e \pmod{p^t}$. If $p = 2$ and $t \geqq 2$, there are two indices e_0 and e_1 corresponding to any given odd residue k mod 2^t such that

$$k \equiv (-1)^{e_0}5^{e_1} \pmod{2^t}, \quad \text{with } e_0 \text{ determined mod } N_0, \quad \text{and } e_1 \text{ mod } N_1, \tag{2}$$

where $N_0 = 2$ and $N_1 = 2^{t-2}$. This extends to $t = 1$ if we choose $N_0 = N_1 = 1$. Index tables exist for prime-powers up to 10,000 from which corresponding values k and e, or k and e_0, e_1 may be read.

Write $N = p_1^{t_1} \cdots p_s^{t_s}$ as a product of powers of distinct primes. If N is even, take $p_1 = 2$ and N_0, N_1 as above; otherwise use (1). Then by the preceding paragraph and the Chinese Remainder Theorem, there is associated with each residue prime to N modulo N a unique index vector $(e_k, e_{k+1}, \cdots, e_s)$ with components

determined respectively to the moduli $N_h, \cdots, N_s$; h is 0 or 1 according as N is even or odd. The group G is isomorphic to the "additive group" of these vectors.

3. The Algorithm. Find the index vector $(u_h, \cdots, u_s)$ of n. Find the g.c.d. d_i of N_i and u_i, and set $K_i = N_i/d_i$, $i = h, \cdots, s$. For each prime p dividing $K_h \cdots K_s$, let $j(=j_p)$ denote the subscript of that one (or of any *one*) of $K_h, \cdots, K_s$ which contains the highest power of p. (We will speak of p as *belonging to* j.) For each i from h to s, write $K_i = L_iQ_i$, where Q_i consists of the prime factors belonging to i. (Thus L_i is prime to Q_i, and Q_i may be 1.) Let S_i denote the set of L_i residues mod K_i obtained by combining each of $0, 1, \cdots, L_i - 1$ with any desired fixed residue mod Q_i. (For example, S_i can be)

$$k, k + Q_i, k + 2Q_i, \cdots, k + (L_i - 1)Q_i, \quad k \text{ a fixed integer.})$$

Then, a set $S(n, N)$ is given by the vectors

$$(3) \quad (e_h + d_hy_h, \cdots, e_s + d_sy_s), \qquad \text{with } y_i \text{ ranging over } S_i,$$
$$\text{and } e_i = 0, 1, \cdots, d_i - 1, \quad (i = h, \cdots, s).$$

Proof. Notice first that the order of the cyclic group generated by n is equal to the least positive integer q such that

$$(4) \qquad qu_i \equiv 0 \pmod{N_i}, \qquad (i = h, \cdots, s),$$

and that $q = Q_h \cdots Q_s$. Also, the number of vectors in (3) is $d_h \cdots d_sL_h \cdots L_s = N_h \cdots N_s/q$. Hence we have only to prove that no two of these vectors are in the same coset. Notice also that the coset of any given vector $(z_h, \cdots, z_s)$ is composed of the q vectors

$$(z_h + xu_h, \cdots, z_s + xu_s), \qquad x = 0, 1, \cdots, q - 1;$$

and for each i, the components $z_i + xu_i$ have a fixed residue modulo d_i. Hence if two vectors in (3), say $(z_h, \cdots, z_s)$ and $(z_h + xu_h, \cdots, z_s + xu_s)$, are in the same coset, they must have the same terms $e_h, \cdots, e_s$ and, for each prime p and the subscript j to which p belongs, $z_j \equiv z_j + xu_j$ modulo the power of p in N_j (since, for every i, $z_i \equiv z_i + xu_i \bmod d_iQ_i$). Hence xu_j is divisible by the power of p in N_j. But (excluding the trivial case where all indices $u_h, \cdots, u_s$ are 0, hence $n = 1$) N_j and qu_j contain the same power of p. Hence xu_j is divisible by the power of p in qu_j, x by the power of p in q (this for every p), x is divisible by q, and the two vectors coincide. Q.E.D.

It remains to convert the vectors in (3) into position numbers mod N. Write $M_i = N/p_i^{t_i}$ $(i = 1, \cdots, s)$. The expression

$$(5) \qquad M_1g_1^{e_1+d_1y_1} + \cdots + M_sg_s^{e_s+d_sy_s},$$

the first term being replaced if N is even by $M_1(-1)^{e_0+d_0y_0}5^{e_1+d_1y_1}$, e_i and y_i having the ranges in (3), gives exactly one leader in every cycle whose position numbers are prime to N. (One can of course read off the values g^{e+dy} from the index table.) Because of the factors M_i the leaders listed in (5) may not be exactly those specified in (3), since the expression in (5) gives the number with g_i to the index $e_i' + e_i + d_i(y_i + y_i')$, where $e_i' + d_iy_i'$ is the index of M_i. But

since e_i' and y_i' are fixed for each i, this has the effect of permuting the residues $e_i \bmod d_i$ and $y_i \bmod L_i$, while keeping y_i still fixed mod Q_i, and thus gives another set $S(n, N)$.

For any divisor v of N, the leaders for the cycles whose position numbers have with N the g.c.d. v can be obtained by multiplying the elements of the set $S(n, N/v)$ by v. The work can be arranged so that all the sets $S(n, N/v)$ for all divisors v of N are constructed in one operation, without duplication of effort. This will be illustrated by an example. The example will also make use of the obvious fact that if N is double an odd, $S(n, N)$ can be derived from $S(n, N/2)$ by simply taking the latter elements and adding $N/2$ to those of them which are even.

4. Example. Let $m = 75$, $n = 73$. Then $N = mn - 1 = 5474 = 2 \cdot 7 \cdot 17 \cdot 23$; 73 is 3^1 mod 7, 3^6 mod 17, 5^4 mod 23.

	p^t	$\phi(p^t)$	u	d	K	Q	L	y	e	Mg^{e+dy}
a)	7	6	1	1	6	3	2	0, 3	0	$391 \cdot 3^x$, $x = e + dy = 0$ or 3
b)	7	6	1	1	6	6	1	0	0	$391 \cdot 3^x$, $x = 0$
	17	16	5	1	16	16	1	0	0	$161 \cdot 3^x$, $x = 0$
	23	22	4	2	11	11	1	0	0, 1	$119 \cdot 5^x$, $x = 0$ or 1.

Notice that row a), with 2 not a factor of Q, is used when 17 is a factor of the modulus, since then the highest power of 2 occurs in $K(17)$. To get the four leaders of primitive cycles mod $7 \cdot 17 \cdot 23$, add either 391 or 2346 to either 119 or 595, and to 161. To get leaders for products by 17 of primitive cycles mod $7 \cdot 23$, add 391 (from row b)) to either 119 or 595—and so forth. The results, with the residues adjusted to be odd as explained earlier, are: 365, 671, 1147, 5363; 3017, 3493; 3247, 3723; 2507, 3289; 391; 161; 119, 595; 2737; also, their doubles, 730, $\cdots$, 1190, 0,—thirty leaders in all.

5. Comments. (i) This technique applies to the determination of leaders of cycles under multiplication by powers of n, in the ring of residues modulo N, if n and N are coprime,—regardless of the matrix interpretation.

(ii) For each prime-power p^t in N, a listing must be given for each power p^s ($s = 1, \cdots, t$), except that 2^1 can be omitted as indicated. In all these, the index u (or the two indices if $p = 2$) need not be changed for $g^u \equiv n \pmod{p^t}$ implies $g^u \equiv n \pmod{p^s}$. Each power p^s should have as many listings as there are different values Q possible for it.

(iii) It is interesting that certain partial sums of the terms in the classical expression (5) give (with appropriate modifications of the exponents) the imprimitive leaders.

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The Fourier Transform of $[c^{2k} + (x - x^{-1})^{2k}]^{-1}$ Arising from Study of Tuned Circuit Spectra

By Charles Walmsley and Arthur S. G. Grant

This Fourier transform is given in tables [1, 2] only in the case when $k = 1$ and $c = 2 \cos \theta$ with $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. The need for it in other cases has arisen in recent work concerned with the study of a random signal obtained by applying a white noise to the input of a band pass filter.

The power transfer function of the filter, and thus the power spectrum of the signal at the filter output, is considered to be of the form

$$T(f) = c^{2k}/[c^{2k} + (f/f_0 - f_0/f)^{2k}]. \tag{1}$$

It should be noted that $T(0) = T(\infty) = 0$, $T(f/f_0) = T(f_0/f)$ and $T(f_0) = 1$. The band-width to the half-power points is obtained by setting $f/f_0 - f_0/f = \pm c$ and solving for the lower and upper cut-off frequencies f_1 and f_2, obtaining

$$f_1/f_0 = [(c^2 + 4)^{1/2} - c]/2 \quad \text{and} \quad f_2/f_0 = [(c^2 + 4)^{1/2} + c]/2. \tag{2}$$

Thus the band-width, $f_2 - f_1$, is equal to cf_0. The shape of the filter response is determined by k, which is assumed to be restricted to positive integral values. The plot of $T(f)$ vs. f on a logarithmic frequency scale is symmetrical about the center frequency f_0 and thus, by a proper choice of c and k, good approximations to practical filter response curves may be obtained. By the Wiener-Khintchine result [3], the auto-correlation function of the filtered noise is given by the Fourier transform of $T(f)$,

$$\phi(\tau) = \int_{-\infty}^{\infty} \frac{c^{2k} \exp(-2\pi j f \tau)\, df}{c^{2k} + (f/f_0 - f_0/f)^{2k}};$$

or, with simple reduction and change of notation,

$$I = \int_{-\infty}^{\infty} \exp(iyx)[c^{2k} + (x - x^{-1})^{2k}]^{-1}\, dx \tag{3}$$

where $c > 0$, $y > 0$, and k is a positive integer.

Contour integration, using the familiar infinite upper semi-circle and Jordan's lemma, shows that the integral I is equal to $2i\pi$ times the sum of the residues of the integrand at the poles in the upper half-plane. The relevant poles are

$$x = \left[c \operatorname{Cis} \frac{2m - 1}{2k}\pi \pm r \operatorname{Cis} \frac{\theta}{2}\right] \Big/ 2, \qquad m = 1, 2, \cdots, k$$

and

$$r^2 \operatorname{Cis} \theta = c^2 \operatorname{Cis} \frac{2m - 1}{k}\pi + 4,$$

where $\operatorname{Cis} \phi (= \exp i\phi)$ denotes $\cos \phi + i \sin \phi$. Evaluation of the residues and algebraic reduction gives the general result:

$$I = (\pi c^{1-2k}/2k) \sum_{m=1}^{k} M, \tag{4}$$

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where

$$M = \exp\alpha\left[\cos\beta\left(\sin\frac{2m-1}{2k}\pi - cr^{-1}\sin\omega\right) + \sin\beta\left(\cos\frac{2m-1}{2k}\pi + cr^{-1}\cos\omega\right)\right]$$

(5)

$$+ \exp\gamma\left[\cos\delta\left(\sin\frac{2m-1}{2k}\pi + cr^{-1}\sin\omega\right) + \sin\delta\left(\cos\frac{2m-1}{2k}\pi - cr^{-1}\cos\omega\right)\right],$$

where α, γ denote

$$\frac{y}{2}\left(-c\sin\frac{2m-1}{2k}\pi \mp r\sin\frac{\theta}{2}\right),$$

β, δ denote

$$\frac{y}{2}\left(c\cos\frac{2m-1}{2k}\pi \pm r\cos\frac{\theta}{2}\right),$$

and

$$\omega = \frac{\theta}{2} - \frac{2m-1}{k}\pi.$$

This is applicable in all cases except when $c = 2$ and k is odd. In this exceptional case the value of M corresponding to $m = \frac{1}{2}(k+1)$ must be replaced by $2(1-y)\exp(-y)$.

In the case when $k = 1$ the result is

$$I = \pi\exp(-\tfrac{1}{2}yc)[c^{-1}\cos\epsilon - (4-c^2)^{-1/2}\sin\epsilon],$$

(6) or $\pi\exp(-\tfrac{1}{2}yc)[c^{-1}\cosh\zeta - (c^2-4)^{-1/2}\sinh\zeta]$,

or $\tfrac{1}{2}\pi(1-y)\exp(-y)$

according as $0 < c < 2$, or $c > 2$, or $c = 2$; where

$$\epsilon = \tfrac{1}{2}y(4-c^2)^{1/2}, \qquad \zeta = \tfrac{1}{2}y(c^2-4)^{1/2}.$$

This result (6), with different notation, agrees with formula 21, §1.2 (p. 9), Vol. 1 of the Bateman Project Tables [1] in so far as that formula is applicable; but it includes also the case (when $c > 2$) not covered by that formula.

For $k = 2$ the result is

(7) $$I = \pi\exp(-yc/2\sqrt{2})[A\exp\phi + B\exp(-\phi)]/(2\sqrt{2}c^3)$$

with $\phi = y(r^2-4)^{1/2}/2\sqrt{2}$, $r^4 = c^4 + 16$,

$A = [1 - cr^{-2}(r^2+4)^{1/2}]\cos(P-Q) - [1 - cr^{-2}(r^2-4)^{1/2}]\sin(P-Q)$,

$B = [1 + cr^{-2}(r^2+4)^{1/2}]\cos(P+Q) + [1 + cr^{-2}(r^2-4)^{1/2}]\sin(P+Q)$,

$P = y(r^2+4)^{1/2}/2\sqrt{2}, \qquad Q = yc/2\sqrt{2}.$

In the degenerate case when $y = 0$ the general result (4), (5), becomes

$$\int_{-\infty}^{\infty} [c^{2k} + (x - x^{-1})^{2k}]^{-1}\, dx = \pi k^{-1} c^{1-2k} \csc(\pi/2k). \tag{8}$$

Special results for $k = 1, c = \frac{1}{2}$ and $k = 2, c = 1/\sqrt{2}$ are

$$\int_0^{\infty} \frac{dx}{\frac{1}{4} + (x - x^{-1})^2} = \int_0^{\infty} \frac{dx}{\frac{1}{4} + (x - x^{-1})^4} = \pi, \tag{9}$$

which may be verified independently.

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On the Evaluation of Certain Complex Elliptic Integrals

By H. A. Lang and D. F. Stevens

1. Introduction. Elliptic integrals of the third kind are occasionally encountered in the form

$$a\Pi(\phi, \alpha^2, k) + \bar{a}\Pi(\phi, \bar{\alpha}^2, k)$$

or

$$a\Pi(\phi, \alpha^2, k) - \bar{a}\Pi(\phi, \bar{\alpha}^2, k)$$

where a, $\bar{a}$ and α^2, $\bar{\alpha}^2$ are complex conjugates and the modulus k is real such that $0 < k^2 < 1.0$. It is usually desirable to rewrite these integrals using only real coefficients and parameters. This paper gives an elementary procedure for the evaluation of these expressions, together with the correction of some existing formulas.

2. Modified Development. We follow the development suggested by Hoüel [1], but use the notation of Byrd and Friedman [2] wherever applicable. Since the modulus k is the same in all of the elliptic integrals considered here, it will be omitted throughout.

For convenience, set

$$2\Pi_1 = \Pi(\phi, \alpha^2) + \Pi(\phi, \bar{\alpha}^2) = \int_0^{\phi} \frac{d\phi}{(1 - \alpha^2 \sin^2\phi)\Delta} + \int_0^{\phi} \frac{d\phi}{(1 - \bar{\alpha}^2 \sin^2\phi)\Delta}$$

$$2i\Pi_2 = \Pi(\phi, \alpha^2) - \Pi(\phi, \bar{\alpha}^2) = \int_0^{\phi} \frac{d\phi}{(1 - \alpha^2 \sin^2\phi)\Delta} - \int_0^{\phi} \frac{d\phi}{(1 - \bar{\alpha}^2 \sin^2\phi)\Delta} \tag{1}$$

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where

$$\Delta = \sqrt{1 - k^2 \sin^2 \phi}$$

Then, if $a = a_1 + ib_1$, we have

$$\begin{aligned} a\Pi(\phi, \alpha^2) + \bar{a}\Pi(\phi, \bar{\alpha}^2) &= 2a_1\Pi_1 - 2b_1\Pi_2 \\ a\Pi(\phi, \alpha^2) - \bar{a}\Pi(\phi, \bar{\alpha}^2) &= 2ia_1\Pi_2 + 2ib_1\Pi_1 . \end{aligned} \tag{2}$$

Π_1 and Π_2 will be given by the equations

$$\begin{aligned} s_1\Pi_1 + t_1\Pi_2 &= -\frac{1}{m_1}F(\phi) - n_1\Pi(\phi, \alpha_1^2) + \tau_1 \\ s_2\Pi_1 + t_2\Pi_2 &= -\frac{1}{m_2}F(\phi) - n_2\Pi(\phi, \alpha_2^2) + \tau_2 \end{aligned} \tag{3}$$

where

$$\tau_i = \int_0^{p_i} \frac{dx}{1 + h_i x^2}, \qquad p_i = \frac{\sin\phi\cos\phi}{(1 + m_i \sin^2\phi)\Delta},$$

and the other subsidiary quantities are obtained as follows. We first set $\alpha^2 = -\gamma_1 - i\gamma_2$, $r^2 = \gamma_1^2 + \gamma_2^2$, $z = \sin^2\phi$ in equations (1) and rationalize:

$$\Pi_1 = \int_0^{\phi} \frac{(1 + \gamma_1 z)\, d\phi}{(1 + 2\gamma_1 z + r^2 z^2)\Delta}$$

$$\Pi_2 = \int_0^{\phi} \frac{-\gamma_2 z\, d\phi}{(1 + 2\gamma_1 z + r^2 z^2)\Delta}$$

We substitute these forms in a general equation (3) to obtain*

$$\int_0^{\phi} \frac{s(1 + \gamma_1 z)\, d\phi}{(1 + 2\gamma_1 z + r^2 z^2)\Delta} - \int_0^{\phi} \frac{t\gamma_2 z\, d\phi}{(1 + 2\gamma_1 z + r^2 z^2)\Delta} + \frac{1}{m}\int_0^{\phi} \frac{d\phi}{\Delta} + n\int_0^{\phi} \frac{d\phi}{(1 - \alpha^2 z)\Delta} = \int_0^{p} \frac{dx}{1 + hx^2} \tag{4}$$

Differentiating both sides of (4) with respect to ϕ we obtain

$$\frac{s(1 + \gamma_1 z) - tz\gamma_2}{(1 + 2\gamma_1 z + r^2 z^2)\Delta} + \frac{1}{m\Delta} + \frac{n}{(1 - \alpha^2 z)\Delta} = \frac{1}{1 + hp^2}\frac{dp}{d\phi}. \tag{5}$$

The left side of this equation reduces to

$$\frac{a_0 + a_1 z + a_2 z^2 + a_3 z^3}{b_0 + b_1 z + b_2 z^2 + b_3 z^3} \cdot \frac{1}{\Delta}$$

where

$$a_0 = s + \frac{1}{m} + n \qquad\qquad b_0 = 1$$

* The two sets of subsidiary quantities arise from the two roots of a quadratic equation in m which results from this general equation. Note that the α in equation (4) is a generic expression for the α_1 and α_2 of equations (3), and is therefore distinct from the α of equations (1) and (2).

$$a_1 = s\gamma_1 - t\gamma_2 - \frac{\alpha^2}{m} + \frac{2\gamma_1}{m} - s\alpha^2 + 2n\gamma_1 \qquad b_1 = 2\gamma_1 - \alpha^2$$

$$a_2 = -s\alpha^2\gamma_1 - t\alpha^2\gamma_2 + nr^2 - \frac{2\gamma_1\alpha^2}{m} + \frac{r^2}{m} \qquad b_2 = r^2 - 2\gamma_1\alpha^2$$

$$a_3 = -\frac{\alpha^2 r^2}{m} \qquad b_3 = -\alpha^2 r^2.$$

The right side of (5) reduces to

$$\frac{c_0 + c_1 z + c_2 z^2 + c_3 z^3}{d_0 + d_1 z + d_2 z^2 + d_3 z^3} \cdot \frac{1}{\Delta}$$

where

$$c_0 = 1 \qquad d_0 = 1$$

$$c_1 = -(2 + m) \qquad d_1 = 2m - k^2 + h$$

$$c_2 = k^2(2m + 1) \qquad d_2 = m^2 - 2mk^2 - h$$

$$c_3 = -k^2 m \qquad d_3 = -k^2 m^2.$$

Setting $a_i = c_i$, $b_i = d_i$ we obtain the following relations:

$$\text{(I)} \quad s + \frac{1}{m} + n = 1$$

$$\text{(II)} \quad s\gamma_1 - t\gamma_2 - \frac{\alpha^2}{m} + \frac{2\gamma_1}{m} - s\alpha^2 + 2n\gamma_1 = -(2 + m)$$

(6)

$$\text{(III)} \quad -s\alpha^2\gamma_1 + t\alpha^2\gamma_2 + nr^2 - \frac{2\gamma_1\alpha^2}{m} + \frac{r^2}{m} = k^2(2m + 1)$$

$$\text{(IV)} \quad \alpha^2 r^2 = m^2 k^2$$

$$\text{(V)} \quad 2\gamma_1 - \alpha^2 = 2m - k^2 + h$$

$$\text{(VI)} \quad r^2 - 2\gamma_1\alpha^2 = m^2 - 2mk^2 - h.$$

The elimination of h from (V) and (VI) and the use of (IV) results in a quadratic in m with real roots:

$$(r^2 + 2k^2\gamma_1 + k^2)m^2 + 2(1 - k^2)r^2 m - r^2(r^2 + 2\gamma_1 + k^2) = 0.$$

Letting m_1, m_2 be these roots, we solve equations (6) for all subsidiary quantities as follows

$$\alpha_i^2 = \frac{m_i^2 k^2}{r^2}$$

$$s_i m_i = m_i - m_i n_i - 1$$

$$n_i m_i = \frac{m_i[\alpha_i^4 - (2 + m_i)\alpha_i^2 + (1 + 2m)k^2] - r^2}{r^2 + 2\gamma_1\alpha_i^2 + \alpha_i^4}$$

$$t_i m_i \gamma_2 = m_i^2 + (\gamma_1 + 2 - \alpha_i^2)m_i + n_i m_i(\gamma_1 + \alpha_i^2) + \gamma_1$$

$$h_i = \frac{(\alpha_i^2 - k^2)(k^4 + 2k^2\gamma_1 + r^2)}{k^2(1 - k^2)}$$

If we take $m_1^2 < m_2^2$ then $0 < \alpha_1^2 < k^2 < \alpha_2^2 < 1$ and $h_1 < 0 < h_2$. In the special case where $r^2 + 2\gamma_1 + k^2 = 0$, we have $m_1 = 0 = \alpha_1^2$ and $h_1 = -r^2$. Then the first equation of (3) is replaced by

$$s_1 \Pi_1 + t_1 \Pi_2 = -\frac{k^2}{r^2} F(\phi) + \frac{1}{r} \tanh^{-1} \frac{r \cos \phi \sin \phi}{\Delta}, \tag{8}$$

the second remaining unchanged.

3. Formulas to be Corrected. It will be seen that the definitions (7) are those given by Byrd and Friedman under formulas 417.00 and 438.00 [2, p. 231, 232, 238] with the exceptions of s_i and t_i. Thus the formulas depending upon these quantities are incorrect. They are 416.00, 417.00, 418.00, 419.00, 437.00, 438.00, 439.00 and 440.00.

We also note that these same errors occurred in Hoüel [1, equations 119, p. LIII]. He gives, in his own notation

$$A = (1 - C)g - 1$$

$$Cg(\mu^2 - 2m\mu \cos \nu + m^2) = g[m^2 + (2 + g)m + (1 + 2m)k^2] - \mu^2.$$

These equations should read respectively

$$Ag = (1 - C)g - 1$$

$$Cg(\mu^2 - 2m\mu \cos \nu + m^2) = g[m^2 + (2 + g)m + (1 + 2g)k^2] - u^2.$$

To correct Byrd and Friedman formulas 416.00, 417.00, 419.00, 437.00, 438.00, and 440.00 it is necessary only to modify the definitions of s_i and t_i to those listed in (7), since as listed they are formal consequences of equations (3) and equation (8). In formulas 418.00 and 439.00, however, we must not only modify s_i and t_i, but also change the sign of the term involving $\Pi(\alpha_i^2, k)$ (in both 418.00 and 439.00), and the sign of the term involving $\tanh^{-1}(rsnu_1cdu_1)$ (in 439.00); these signs should all be negative. (See the end of the next section for a modified, corrected version of 439.00.)

4. Simplication of Results. The form of definitions (7) suggests the following simplifications in the Byrd and Friedman formulas under consideration. Let us multiply equations (3) by m_i to obtain

$$s_i\Pi_1 + t_i\Pi_2 = -F(\phi) - n_i\Pi(\phi, \alpha_i^2) + \tau_i \qquad i = 1, 2$$

where now

$$\tau_i = \int_0^{p_i} \frac{m_i\, dx}{1 + h_i x^2},$$

the p_i are defined as before, and definitions (7) are replaced by:

$$\sigma_i^2 = \frac{m_i^2 k^2}{r^2}$$

$$s_i = m_i - n_i - 1$$

$$n_i(r^2 + 2\gamma_1\alpha_i^2 + \alpha_i^4) = m_i[\alpha_i^4 - (2 + m_i)\alpha_i^2 + (1 + 2m)k^2] - r^2$$

$$t_i\gamma_2 = m_i^2 + m_i(\gamma_1 + 2 - \alpha_i^2) + n_i(\gamma_1 + \alpha_i^2) + \gamma_1$$

$$h_i k^2(1 - k^2) = (\alpha_i^2 - k^2)(k^4 + 2k^2\gamma_1 + r^2).$$

This permits us to write formulas 416.00, 417.00, 437.00 and 438.00 with no explicit appearances of m_1 or m_2. For example, 416.00 becomes:

$$416.00': \; a\Pi(\alpha^2) + \bar{a}\Pi(\bar{\alpha}^2) = \frac{2}{s_1 t_2 - s_2 t_1}\{[a_1(t_1 - t_2) + b_1(s_1 - s_2)]K$$

$$+ n_2(a_1 t_1 + b_1 s_1)\Pi(\alpha_2^2) - n_1(a_1 t_2 + b_1 s_2)\Pi(\alpha_1^2)\}$$

A similar simplification is possible in the special case when $m_1 = 0 = \alpha_1^2$. In this case we multiply the second equation of (3) by m_2 as above, and multiply equation (8) by r^2 to obtain

$$s_1 \Pi_1 + t_1 \Pi_2 = -k^2 F(\phi) + r \tanh^{-1} \frac{r \cos\phi \sin\phi}{\Delta}$$

where

$$s_1 = r^2 - k^2$$

$$t_1\gamma_2 = 2r^2 + r^2\gamma_1 + k^2$$

(n_2, s_2 and t_2 are defined as in (7').) This permits us to write formulas 418.00, 419.00, 439.00, and 440.00 with no explicit appearances of m_2 or r_2 (But note that the two occurrences of r in formulas 439.00 and 440.00 are preserved.) For example, 439.00 (with the sign changes mentioned in Section 4 incorporated) becomes

$$439.00': \; a\Pi(u_1, \alpha^2) + \bar{a}\Pi(u_1, \bar{\alpha}^2) = \frac{2}{s_2 t_1 - t_2 s_1}\Big\{[a_1(k^2 t_2 - t_1) + b_1(k^2 s_2 - s_1)]u_1$$

$$- n_2(a_1 t_1 + b_1 s_1)\Pi(u_1, \alpha_2^2) + (a_1 t_1 + b_1 s_1)\tau^2$$

$$- r(a_1 t_2 + b_1 s_2) \tanh^{-1} \frac{r \cos\phi \sin\phi}{\Delta}\Big\}.$$

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The Numerical Evaluation of the Eighteenth Perfect Number

By D. Scheffler and R. Ondrejka

On November 17, 1959 the IBM 709 installation at the National Aviation Facilities Experimental Center in Atlantic City, New Jersey computed the largest known perfect number, corresponding to the eighteenth Mersenne prime [1]. The result was checked by recomputation one week later. Running time for this cal-

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culation was approximately five minutes. This perfect number has 6433 proper divisors, and its decimal representation is as follows:

$2^{2216}(2^{2217} - 1) =$

33 57083 21319 86724 43701 08772 11080 38484
11380 28499 87972 54549 96241 57348 21584 50444 04288 20487
78809 43769 03884 49535 77426 08498 85573 69475 99061 73841
15743 84247 30130 80704 76236 55942 23617 48505 09108 53782
76585 90642 32548 24947 61473 19657 90746 56099 91860 07644
04702 18166 02944 69121 77873 79658 22199 90166 34780 93006
07502 23592 23201 84998 56361 44177 18592 54020 78185 07301
50450 97727 08485 94647 43635 53778 15002 84915 88024 48863
06461 78598 29560 72060 01347 49556 17851 48168 01859 88557

13660 92248 41817 87708 36089 51191 12317 48852 26416 13068
31977 10667 39235 10073 74503 75540 33525 31476 22794 35900
71651 70269 75942 41031 95552 98989 71218 00121 46417 74673
13494 44715 62560 95717 96578 81556 41912 21029 35450 29975
18133 40515 17095 61679 51095 45364 94855 76150 66010 16891
60658 01177 01932 74226 30828 05077 86835 04954 91125 76654
51011 96704 56745 93989 01942 05255 17538 44844 89909 32896
76469 88163 15598 24715 64998 19626 16327 51283 12787 95091
98074 25319 34095 80454 56248 86643 83465 37988 50027 35506
15398 88515 06645 13775 92755 53988 21942 54397 64732 39982

47124 38125 05411 75238 37438 25674 44370 55019 44105 10064
89972 34160 91179 78404 56379 49920 04873 05751 84557 48701
44495 12383 77139 62049 42879 82489 52982 72331 40637 01483
74088 56156 19951 54576 69607 96405 21269 08149 26560 17860
94447 59556 04400 59050 09176 35471 14092 25537 13974 25807
86755 43521 12542 19478 48154 94784 27620 11708 45949 27467
46329 85210 42107 55317 84918 35892 66903 95463 64972 14522
65405 71348 43880 43911 63448 54323 58638 80664 53138 26206
59113 12662 32422 00783 55773 45584 22572 03105 18698 14337
67362 19283 02111 92876 17896 14688 55848 60065 04887 63157

01088 79621 95936 40826 31162 22733 28035 60330 94756 42390
80449 94601 56797 85536 10182 46696 10125 39222 54567 24090
83153 85468 24093 18461 66962 49598 34076 07141 60125 18895
44407 00881 58747 44654 76950 72686 78051 75774 69568 91212
48545 62611 21386 66740 77111 39619 07153 09233 55823 17866
27053 74393 03504 90226 03882 47974 23347 99407 13028 01487
69298 59774 37781 93050 34874 97407 86928 09603 39062 95910
19923 81813 38557 85697 81918 60647 25620 97081 68229 11615
63009 78059 19702 68557 26877 64976 70726 84960 46345 27631
60384 09383 82922 77544 91185 78596 58328 88833 26285 25056.

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1. H. RIESEL, "A new Mersenne prime," *MTAC*, v. 12, 1958, p. 60.

A Note on Gaussian Twin Primes

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If $m^2 + 1$ is a prime, then $m + i$ is a Gaussian prime and conversely. If $(n-1)^2 + 1$ and $(n+1)^2 + 1$ are both prime, then $n - 1 + i$ and $n + 1 + i$ form a pair of Gaussian twin primes [1, p. 82]. This is the case for $n = 3, 5, 15, 25, 55, \cdots, 184705, 184745, 184755, \cdots$; the corresponding (rational) primes being 5 and 17 for $n = 3$, and 34134040517 and 34134779537 for $n = 184755$. Let $g(N)$ be the number of such pairs for $4 \leqq n + 1 \leqq N$.

Similarly, let $z(N)$ be the number of pairs of rational twin primes, $n - 1$ and $n + 1$, (such as $n = 4, 6, 12, 18, \cdots$), for $5 \leqq n + 1 \leqq N$. Hardy and Littlewood [2] conjectured that

$$z(N) \sim 1.32032 \int_2^N \frac{dn}{(\log n)^2} \tag{1}$$

where

$$1.32032 \cdots = 2 \prod_{p=3}^{\infty} \left(1 - \frac{1}{(p-1)^2}\right), \tag{2}$$

the product being taken over all odd primes.

By the use of a sieve argument very similar to that recently presented [3] in support of another Hardy-Littlewood conjecture, the following asymptotic relation was obtained:

$$g(N) \sim 0.36932 z(N), \tag{3}$$

where

$$0.36932 \cdots = \frac{1}{4} \prod_{p=3}^{\infty} \left[1 - 2\left(\frac{-1}{p}\right)(p-2)^{-1}\right]. \tag{4}$$

Here $(-1/p)$ is the Legendre symbol. Assuming the truth of both (3) and (1), we have

$$g(N) \sim 0.48762 \int_2^N \frac{dn}{(\log n)^2}. \tag{5}$$

We may compute the constant in (5), and therefore also that in (4), from

$$0.48762 \cdots = \frac{\pi^2}{8} \prod_{p=4m+1} \left(1 - \frac{4}{p}\right)\left(\frac{p+1}{p-1}\right)^2, \tag{6}$$

the product being taken over all primes of the form $4m + 1$. The evaluation of the right side of (6) is facilitated by a transformation similar to that previously used [3] in computing the Hardy-Littlewood constants, h_s.

The number of Gaussian twin pairs, $g(N)$, was determined for $N = 500(500)185000$, by counting these pairs in a recently computed table [1, p. 81] of the greatest prime factor of $n^2 + 1$ for $n = 1(1)185000$. A short summary is shown in Table 1 together with a comparison of $g(N)$ and

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TABLE 1
Gaussian and Rational Twin Primes

N	$g(N)$	$\bar{g}(N)$	$g/\bar{g}$	$z(N)$	$\bar{z}(N)$	$z/\bar{z}$
10000	76	79.1	0.961	205	214.2	0.957
20000	127	132.1	0.961	342	357.8	0.956
30000	180	179.8	1.001	467	486.7	0.959
40000	234	224.3	1.043	591	607.4	0.973
50000	276	266.8	1.034	705	722.5	0.976
60000	321	307.8	1.043	811	833.4	0.973
70000	361	347.5	1.039	905	940.9	0.962
80000	403	386.2	1.044	1007	1045.7	0.963
90000	437	424.0	1.031	1116	1148.2	0.972
100000	463	461.2	1.004	1224	1248.7	0.980
110000	502	497.7	1.009			
120000	532	533.6	0.997			
130000	568	569.0	0.998			
140000	598	603.9	0.990			
150000	629	638.4	0.985			
160000	660	672.6	0.981			
170000	696	706.4	0.985			
180000	734	739.8	0.992			
185000	762	756.4	1.007			

$$\bar{g}(N) = 0.48762 \int_2^N \frac{dn}{(\log n)^2}. \tag{7}$$

Also shown are Glaisher's counts [4] of $z(N)$ to $N = 10^5$ and

$$\bar{z}(N) = 1.32032 \int_2^N \frac{dn}{(\log n)^2}. \tag{8}$$

In this range of N the deviations from unity of $g(N)/\bar{g}(N)$ and $z(N)/\bar{z}(N)$ are about equal in magnitude, [5].

The slow oscillations of $g(N)/\bar{g}(N)$ around one have two significant consequences.

1. They make improbable any value of the *constant* in (5) which differs more than slightly from the theoretical value, (6).

2. They make possible a sensitive test for the correctness of the *function of N* assigned to the proposed asymptote, $\int_2^N dn/(\log n)^2$. For since $| g(N) - \bar{g}(N) | \ll \bar{g}(N)$, even small functional modifications in $\bar{g}(N)$ would greatly alter the phase, frequency, and amplitude of the corresponding oscillations of $g(N)/\bar{g}(N)$ around one. Now consider $P(N)$, the *total* number of Gaussian primes of the form $m + i$, [1, see Table on p. 78; p. 81]. The corresponding Hardy-Littlewood conjecture reads

$$P(N) \sim \bar{P}(N) = 0.68641 \int_2^N dn/\log n, \tag{9}$$

and similar remarks are applicable to the function $\bar{P}(N)$ and to any oscillations of $P(N)/\bar{P}(N)$. But if (9) and (5) are both valid, we must expect that any slow oscillations of $g(N)/\bar{g}(N)$ and $P(N)/\bar{P}(N)$ will agree in phase and frequency. For, where

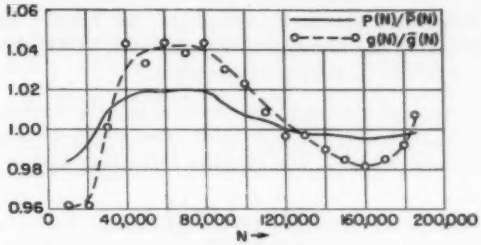

FIG. 1—Comparison of $g(N)/\bar{g}(N)$ with $P(N)/\bar{P}(N)$.

there is an excess of primes, there should generally also be an excess of twins, and if the oscillations are slow, then any complicating higher frequency fluctuations in the local density will largely disappear by integration. In Figure 1 we compare graphs of $g(N)/\bar{g}(N)$ and $P(N)/\bar{P}(N)$ for N to 185000. Very close agreement is seen in the phase and frequency of the slow oscillations. Since such an agreement would seem improbable if either or both of (5) and (9) were false, it may be regarded as providing further evidence in their favor.

The difficulties that stand in the way of a proof of (3), (assuming it to be true) are similar to those previously discussed for other problems [3]. Thus it is unlikely that (3) will be proven without a simultaneous solution of the long outstanding Goldbach, twin prime, and $n^2 + 1$ prime problems.

In conclusion it should be noted that the Gaussian twins on the line $n + i$ are by no means the only "twins" in the Gauss plane. On the line $n + 2i$, for instance, we not only have twins, $n = (179983, 179985)$, and triplets, $n = (423, 425, 427)$, but even one octuplet, $n = (-7, -5, -3, -1, +1, +3, +5, +7)$.

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1. DANIEL SHANKS, "A sieve method for factoring numbers of the form $n^2 + 1$," *MTAC*, v. 13, 1959, p. 78–86.
2. G. H. HARDY & J. E. LITTLEWOOD, "Partitio numerorum III: On the expression of a number as a sum of primes," *Acta. Math.*, v. 44, 1923, p. 42.
3. DANIEL SHANKS, "On the conjecture of Hardy and Littlewood concerning the number of primes of the form $n^2 + a$," *Notices*, Amer. Math. Soc., v. 6, 1959, p. 417. Abstract 559–52. A forthcoming paper with the same title will give an expanded version of this report.
4. J. W. L. GLAISHER, "An enumeration of prime-pairs," *Messenger Math.*, v. 8, 1878, p. 28–33.
5. The empirical evidence for (1) is much more extensive. D. H. Lehmer has computed $z(N) = 183728$, $\bar{z}(N) = 183582$, and $z/z(N) = 1.0008$ for $N = 37 \cdot 10^6$. See the review, UMT 3, of D. H. LEHMER, "Tables concerning the distribution of primes up to 37 million," *MTAC*, v. 13, 1959, p. 56.

29[B].—GEORGE E. REYNOLDS, *A New Method of Cube Root Extraction on Desk Calculators*, Electronics Research Directorate, Air Force Cambridge Research Center, Air Research and Development Command, USAF, April 1958, (ASTIA Doc. No. AD 133760), vi + 27 p., multilithed.

The method consists of the use of one iteration of a third-order iterative process for computing $\sqrt[3]{N}$, with the aid of a table from which certain functions of an approximate root A can be read. The iterative formula is used in the form

$$\frac{(2A)\left(\frac{N}{A^3}+\frac{1}{2}\right)}{\frac{N}{A^3}+2},$$

both $2A$ and A^3 being read from a table. Thus only one multiplication and two divisions, plus some minor auxiliary operations, are necessary.

Three tables are provided, these being so-called five-place, six-place, and seven-place cube root tables. The author states that a maximum error of one unit in the last significant digit of the rounded answer can be achieved. A few spot checks have revealed no cases in which the statement is false. It is worth noting that the only instance discovered by the reviewer in which the result obtained from the tables differed from the correct result by one unit in the last digit occurred in the beginning of the five-place table for the range in which the tabulated value of $2A$ has 5S instead of 6S. This agrees with the author's statements on p. 11–12 concerning the location of the maximum error in the tabulated values.

The instructions as to procedure are sufficiently explicit except in respect to the number of guard figures to be kept in the quotient N/A^3. One may infer from the corrected version (furnished with the review copy as an erratum) of the example on p. 15 that the author believes one guard figure will suffice, although there is no consideration of this issue in the paper.

Equation (21) is incorrect. Its right side should read

$$(2A)(N/A^3 + \tfrac{1}{2}) \div (N/A^3 + 2).$$

The reviewer is allergic to the use (p. 9) of the colloquial phrase "several times greater than" in lieu of "several times as large as". There is no such thing (p. 13, line 3 *fb*) as *the* Newton-Raphson Method for the cube root, since there are many equivalent equations with the cube root of N as a solution. The reviewer is unfamiliar with the term (p. 14, line 8) "increasing asymptotically". Is "monotonically" the adverb intended?

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30[G].—JOHN L. SELFRIDGE, *On Finite Semigroups*, Dissertation, University of California, Los Angeles, multilithed typescript, iv + 185 pp.

A *system* is a set of abstract elements, together with a binary operation [in this review called multiplication] defined from the cartesian product $S \times S$ to a set V. For $a, b \in S$, the value of the product is written ab. The system is *closed* whenever $V \subset S$. The *order* of the system is the number of elements in S.

Two systems S and T with sets of values U and V are called *isomorphic* whenever there is a one-to-one function φ from $S \cup U$ to $T \cup V$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in S$. They are *anti-isomorphic* whenever $\varphi(ab) = \varphi(b)\varphi(a)$ for all $a, b \in S$. "Using isomorphism as an equivalence relation $\cdots$ [systems] are divided into *classes*, all $\cdots$ [systems] in a class being isomorphic *copies*. If a pair of classes are anti-isomorphic (i.e., they have anti-isomorphic representatives) they are called a *type*. The remaining classes are types which are anti-isomorphic to themselves" (p. 3).

If a system is closed and multiplication is associative, it is called a *semigroup*. A semigroup in which, for some p, all products of p elements are equal is called *nilpotent*. The classification and enumeration of general systems of finite order is an exceedingly difficult problem, unsolved even for finite groups. In 1954 the reviewer (with Dr. Selfridge's able assistance) wrote a code [1] which caused SWAC to punch multiplication tables for all semigroups of order 4. One of the main tasks was to arrange the 3492 tables by types, and to select one table to represent each type. In order to represent the tables compactly and deal with them arithmetically, the elements of the semigroup were given the names 0, 1, 2, 3, and the sixteen entries in a multiplication table were written row after row in a single line of 16 base 4 digits. It was convenient to select as the representative of a type of semigroup that one whose table comes first in the lexicographic order by rows. Such a multiplication table is called a *row-normal table* by Selfridge. Similarly, *column-normal table* is defined.

The above considerations were purely matters of machine convenience, devised for a limited purpose. In his dissertation Selfridge has made a systematic study of multiplication tables, both for semigroups and for more general systems, and through an analysis of tables and row-normal tables has obtained new information about the structure of the algebraic systems. This is a good illustration of the feedback from digital computation to pure mathematics. Perhaps the greatest benefit of automatic computation for any problem is the resulting increased theoretical comprehension of the problem.

In the dissertation definitions and elementary properties of multiplication tables occupy p. 1–7. For example, it is proved that there are at least $(n^2 + n)^q/(n!)(n^2!)$ types of commutative systems of order n, where $q = n(n + 1)/2$. On p. 8–14 the author enumerates all possible initial rows of a row-normal semi-group table, and proves that there is an example of a semigroup corresponding to each possibility. Certain last rows are also enumerated.

Pages 15–19 contain a classification and enumeration of all tables corresponding to nilpotent semigroups in which every product of three elements if 0. As a result, it is proved that there are exactly Σ_n nilpotent semigroups of order n in which every triple product is 0, where

$$\Sigma_n = \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} (k-i)^{(n-k)^2}.$$

The author then tabulates Σ_n and certain auxiliary quantities for $n = 1(1)7$. It is startling to find that $\Sigma_7 = 5{,}944{,}080{,}072$, a low lower bound for the number of semigroups of order 7.

Pages 20–22 list the results of the enumeration of certain algebraic systems of

orders 1, 2, 3, 4, and 5. The four properties of closure, associativity, commutativity, and having a multiplicative unit are considered separately, and the list gives the number of types, classes, and tables. Some of the numbers come from the earlier theory, some come from a list of semigroups of order 5, and some apparently come from other enumerations. We find, for example, that there are 183,732 semigroups of order 5, of which (by looking at Σ_5) we see that only 11,725 are of the types considered earlier. We find also that there are 720 types of closed commutatives systems of order 4 which have a unit.

After a three-page history of the use of high-speed computers to seek and enumerate various systems, the author devotes p. 26–33 to indicating how in 1955 he and Professor T. S. Motzkin found and classified all semigroups of order 5 on SWAC, using only 256 cells of rapid-access storage. To reduce the time, only normal tables were computed. There follows a two-page study of the feasibility of carrying on to order 6. Exclusive of punch time, on SWAC the principal search code took 3 minutes for order 4, 40 minutes for order 5, and might be expected to take about 25 hours for order 6. In all cases the editing and preparation of the tables for publication remains the major part of the work.

The greatest bulk of the dissertation is the appendix on pp. 41–185, where the semigroups of order 5 are tabulated in the form of multiplication tables. It is difficult (though not impossible) for the reader to determine just what is listed; the explanation (p. 28 and 29) refers to lists L_1 and L_2, which are not the lists in the appendix! It is to be hoped that Dr. Selfridge will add an explanation to any existing copies of this appendix.

Here is the reviewer's explanation of the appendix, based partly on personal copies of L_1 and L_2. There are 1915 distinct classes of these semigroups. Consider each multiplication table to be in row-normal form, and the resulting 1915 tables to be given *class numbers* [reviewer's term] from 0 to 1914 in lexicographic order by rows. Now 405 of the 1915 semigroups (including the 325 commutative semigroups) have the property that each is anti-isomorphic to itself, while the other 1510 semigroups form 755 anti-isomorphic pairs. The author omits the table in each anti-isomorphic pair that has the larger class number, and thus obtains an ordered set of 1160 row-normal tables, representing each of the 1160 distinct types of semigroups of order 5. These are given *type numbers* [reviewer's term] from 0 to 1159 in lexicographic order by rows.

The appendix lists each of these 1160 semigroups in the order of the type numbers. For each semigroup the following five items are given across the page, as listed by an IBM tabulator on multilith masters:

1) The type number.
2) The multiplication table in row-normal form (a 5-by-5 array of digits 0 through 4).
3) The class number.

That is all the information for the commutative semigroups. For the non-commutative semigroups, there are two more pieces of information:

4) The class number of the semigroup to which the listed semigroup is anti-isomorphic. (This is omitted if the class number is the same as item 3. That a noncommutative semigroup can be anti-isomorphic to itself had not been observed by all workers with semigroups.)

5) The column-normal isomorph of the same multiplication table (omitted if the row-normal table is itself column-normal). The semigroup is anti-isomorphic to itself if and only if item 4 is omitted and item 5 is included.

There is a bibliography of 34 titles, including work of a Japanese group including Professor Takayuki Tamura. It is notable that the Japanese group, working by hand, obtained all 126 semigroup types of order 4 prior to SWAC (without error), and that they finished obtaining the semigroups of order 5 almost simultaneously with the American group of Motzkin, Selfridge, and SWAC (but with at least one error discovered by the American group). It appears that in computing semigroups it has been a reasonably fair match between Japanese without an abacus and Americans armed with an electronic digital computer! The text of the dissertation reviewed here does not refer to the Japanese computation for order 5.

Minor criticisms: On page 8, line 8, for ". . . an element of a semigroup . . ." read ". . . an element a of a semigroup . . . ". In the footnote on page 8, for m read a. In reference [A2] the authors' names are permuted. In this and other references the authors' first names have been carefully omitted [why is this done so often?].

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1. GEORGE E. FORSYTHE, "SWAC computes 126 distinct semigroups of order 4," Amer. Math. Soc., *Proc.*, v. 6, 1955, p. 443–447.

31[K].—E. L. LEHMANN, *Testing Statistical Hypotheses*, John Wiley & Sons, Inc., New York, 1959, xiii + 369 p., 24 cm. Price $11.00.

This long-awaited book is a welcome addition to Wiley's fine series of texts in the different areas of modern mathematical statistics, and the author is to be thanked and congratulated for a difficult but needed job exceedingly well done. The title does not convey an adequate idea of the scope of the book, which includes material from many branches of statistics, nor of the exceedingly helpful devices used that make it possible to catch up with the most recent advances. Going through it is a refreshing and stimulating experience.

The book provides "a systematic account of the theory of hypothesis testing and of the closely related theory of estimation by confidence sets. The principal applications of these theories are given, including the one- and two-sample problems concerning normal, binomial and Poisson distributions. There is also a treatment of permutation tests and of some aspects of the analysis of variance and of regression analysis. Introductions to multivariate and sequential analysis, and to non-parametric tests are offered. Methods based on large sample considerations (χ^2 and likelihood ratio tests) are sketched. The emphasis throughout is on the various optimum properties of the procedures. These are discussed in terms of the Neyman-Pearson formulation, but against a background of decision theory which frequently permits a broader justification of the results."

The level of the treatment is set by the fact that "the natural framework for a systematic treatment of hypothesis testing is the theory of measure in abstract spaces." By unrestricted use of the abstract approach, the author is enabled to bring the prepared reader abreast of the very latest developments. For this, one should have an appreciation, if not a knowledge, of concepts in measure theory,

as well as a considerable background in "classical" mathematical statistics. It should be stated, however, that all necessary theorems are stated and discussed, if not always proved.

The text is greatly enriched by some very valuable time-saving features, such as footnotes that immediately relate an outside source to the point under discussion, and annotated lists of references, appearing at the close of each chapter and frequently summarizing a paper in a single sentence, thereby giving the reader a bird's-eye view of the latest pertinent papers as well as of the earlier literature. Furthermore, for each section a set of substantial exercises, totalling over 200, is provided. Many of these are accompanied with outlines of solutions, and provide introductions to additional topics.

Attention to estimation as a special case of hypothesis testing is essentially limited to confidence sets, point estimation receiving very little consideration. For the sake of completeness one would like to have seen some treatment of the Cramèr-Rao inequality and its modern development by Bhattacharya and others, as well as minimum-variance estimation on which so much practical work is based. This, however, is hardly a criticism, since treatment of estimation is not a main purpose.

While not designed as a "cookbook" in the analysis of actual data, because of its advanced nature the book does give a deep understanding of the many tests and their relationships to a unified theory. As such, and in view of its time-saving features, the book is well worth the price, and should be in the possession of the advanced worker in mathematical statistics and others having the requisite background.

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32[L].—L. N. KARMAZINA & É. A. ČHISTOVA, *Tablítsy funktsiĭ Besselía ot mnimogo argumenta i integralov ot nikh* (*Tables of Bessel functions of imaginary argument and of integrals involving them*). Izdatel'stvo Akademii Nauk SSSR (Press of the Academy of Sciences of the USSR), Moscow, 1958, 328 p., 27 cm. Price 37 rubles 15 kopecks.

This volume in the series of Mathematical Tables from the Computational Center of the Academy of Sciences is a continuation of earlier work [1] on Bessel functions of real argument.

The present tables were prepared on the electronic computer STRELA. In the main table (pages 19–328), the values of the following seven functions are given for $x = 0(.001)5(.005)15(.01)100$:

$$e^{-x}I_0(x), \qquad e^{-x}I_1(x), \qquad e^{-x}\int_0^x I_0(u)\,du,$$

$$e^{x}K_0(x), \qquad e^{x}K_1(x), \qquad e^{x}\int_x^\infty K_0(u)\,du, \quad \text{and} \quad e^{x}.$$

The values are given here to 7D except near the origin, where they are to 7S; no differences are given. Near the origin, the following auxiliary functions are given:

$$I_0(x) \quad \text{and} \quad E_0(x) = K_0(x) + \ln x\, I_0(x)$$

for $x = 0(.001).15$ to 7S and 7D, respectively;

$$I_1(x) \quad \text{and} \quad E_1(x) = x[K_1(x) - \ln x\, I_1(x)]$$

for $x = 0(.001).2$ to 7D. By means of the formulas given on page 11, a number of related integrals can be evaluated by using the present tables.

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1. É. A. CHistova, *Tablitsy funktsiĭ Besselia ot deĭstvitel'nogo argumenta i integralov ot nikh (Tables of Bessel functions of real argument and of integrals involving them)*. Izdatel'stvo Akademii Nauk SSSR (Press of the Academy of Sciences of the USSR), Moscow, 1958. *Math. Comp.*, v. 14, 1960, p. 79–80.

33[L].—F. A. PAXTON & J. E. ROLLIN, *Tables of the Incomplete Elliptic Integrals of the First and Third Kind*, Curtiss-Wright Corporation, Research Division, Quehanna, Pennsylvania, June 1959, 436 p., 28 cm.

This large table gives values of the elliptic integral of the third kind, which in the notation of the authors is

$$\Pi(\phi, \alpha^2, k) = \int_0^{\phi} \frac{d\theta}{(1 - \alpha^2 \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}}$$

including, as a special case, the elliptic integral of the first kind, $F(\phi, k) = \Pi(\phi, 0, k)$. Values are tabulated to 7D without differences for $\phi = 0(1°)90°$, $\alpha^2 = 0(.02)1$, $k^2 = 0(.02)1$. The values were computed on an IBM 704 by Simpson's rule. The authors appear to claim general accuracy within 10 final units, except possibly for $\sin \phi$, α^2, and k^2 near unity. The reviewer has encountered nothing which invalidates the general claim, but the exception is certainly to be noticed.

There are several ways in which certain values, especially of the complete integrals ($\phi = 90°$), may be checked from existing tables with little or no arithmetic. Using for brevity the references of *MTAC*, v. 3, 1948, p. 250, let us consider four checks:

(i) Values of $F(90°, k) = K$ with argument k^2 are given to 10–12D in Hayashi 1. They show that the values of Paxton and Rollin are systematically too small; the error rises from 4 final units at $k^2 = .02$ to 11 final units at $k^2 = .98$. These errors are practically within the claimed limits. The machine value at $k^2 = 0$, which should equal $\frac{1}{2}\pi$, is 6 final units too small, but has been corrected by hand.

(ii) Values of $F(\phi, k)$ for $\phi = 0(1°)90°$, $k^2 = \frac{1}{2}$ (modular angle = 45°) are given to 10D in Legendre 3, 5, 6, 7, 8 (also to 12D in Legendre 3, 5). They show no errors in Paxton and Rollin exceeding 5 final units.

(iii) Values of

$$\Pi(\phi, 0, 1) = \int_0^{\phi} \sec \phi \, d\phi,$$

the inverse gudermannian, are given to 9D in Legendre 3, 5, 6, 7, 8 (also to 12D in Legendre 3, 5). They show that the later values of Paxton and Rollin are systematically too small, for example by about 1, 11, 45, 141 final units at $\phi = 45°$,

85°, 88°, 89°, respectively. It may be added that values of $F(\phi, k)$ for $\phi = 0(1°)90°$, $k^2 = 0(.01)1$ are given in Samoĭlova-Ĭakhontova 1, but to only 5D.

(iv) When $\alpha^2 = k^2$, we have

$$\Pi(90°, \alpha^2, k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{-3/2}\, d\theta,$$

which is known to equal $E/(1 - k^2)$, where E is the complete elliptic integral of the second kind, given to 10D with argument k^2 in Hayashi 3. Evaluation shows that the values of Paxton and Rollin are systematically too small, for example by 4, 15, 31, 177, and almost 590 units at $k^2 = .2, .8, .9, .96, .98$, respectively.

As far as the reviewer's examination has gone, it seems likely that the table is correct everywhere to about 4D, almost everywhere to 5D, and in large regions to 6D and 7D. Even with this limitation, the table (which, as far as the reviewer is aware, is a corporation research report rather than a published work) must be regarded as epoch-making in the history of the tabulation of the elliptic integral of the third kind. It should be added that another sizable table of this integral, by Selfridge and Maxfield [1], appeared in 1958, but with a different argument system.

A. F.

1. R. G. Selfridge & J. E. Maxfield, *A Table of the Incomplete Elliptic Integral of the Third Kind*, Dover Publications, New York, 1958 (also Constable, London).

34[P, W].—Armour Research Foundation, *Proceedings of the Fourth Annual Computer Applications Symposium*, 1957, sponsored by the Armour Research Foundation of Illinois Institute of Technology, 1958, x + 126 p., 23 cm. Price $3.00.

This is a collection of 15 papers based on 12 talks, two luncheon addresses and a panel discussion. Practically all the symposium concerned itself with applications of digital computers.

The program of the symposium covered two days, one devoted to a session on "Business and Management Applications," the other to a session on "Engineering and Research Applications."

The following seven papers concerned with the first subject appear in the *Proceedings*:

An Extensive Hospital and Surgical Insurance Record-Keeping System—R. J. Koch,

A Central Computer Installation as a Part of an Air-Line Reservations System—R. A. McAvoy,

Fitting a Computer into an Inventory-Control Problem—O. A. Kral,

The Problems of Planning New Metropolitan Transportation Facilities and Some Computer Applications—J. D. Carroll, Jr.,

Data-Processing Tasks for the 1960 Census—D. H. Heiser & Dorothy P. Armstrong,

The Handling of Retail Requisitions from a General Warehouse—M. J. Stoughton,

Automatic Programming for Business Applications—Grace M. Hopper.

The papers generated at the session on "Engineering and Research Applications" are as follows:

Digital Simulation of Active Air Defense Systems—R. P. Rich,

Statistical Calculations in Product-Development Research—E. B. Gasser,

Progress in Computer Application to Electrical Machine and System Design—E. L. Harder,

How Lazy Can You Get?—A. L. Samuel,

The Solution of Certain Problems Occurring in the Study of Fluid Flow—L. U. Albers,

A Dual-Use Digital Computer for Dynamic System Analysis—E. H. Clamons & R. D. Adams,

The Status of Automatic Programming for Scientific Problems—R. W. Bemer, Panel Discussion.

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35[S].—Val J. Ashby & Henry C. Catron, *Tables of Nuclear Reaction Q Values*, UCRL-5419, Lawrence Radiation Laboratory, University of California, Livermore, California, 1959, 330 p., 28 cm. Price $5.00. Available from the Office of Technical Services, Department of Commerce, Washington 25, D. C.

Tables of nuclear reaction Q values have been calculated from nuclide masses, when possible, for those 42 reactions involving γ, n, p, d, t, He^3, or He^4 as either incident or product particle for about 650 target nuclides. Approximately 8000 Q values are tabulated.

Authors' Summary

36[S, T].—James Miller, John M. Gerhauser & F. A. Matsen, *Quantum Chemistry Integrals and Tables*, University of Texas Press, Austin, Texas, 1959, 1224 p., 26 cm. Price $15.00.

One of the major difficulties in making quantum-mechanical calculations of the properties of atoms and molecules is the evaluation of the large number of difficult integrals which appear. This volume contains tables for the evaluation of the one- and two-center $1s$, $2s$, and $2p_\sigma$ integrals involved in energy and dipole moment calculations. No $2p\pi$ integrals are included. The tables are based on the usual Slater-type atomic orbitals. Molecular integrals are not tabulated directly, but rather auxiliary functions (A, B, G, and W in the usual Kotani notation). Some computation is therefore still necessary to arrive at a desired molecular integral, but it is within the reach of a desk calculator.

The tables were computed on an IBM 650 and reproduced by a photo-offset process to avoid introduction of errors. They appear quite clear and legible. The short textual parts of the book contain formulas as well as recommended interpolation procedures and their expected accuracy—a most welcome feature. Over 90 per cent of the pages are devoted to the difficult W functions.

The present tables naturally invite comparison with previous tables of molecular integrals, particularly those by Kotani, Amemiya, Ishiguro, and Kimura

[1], and by Preuss [2]. These both contain a wider variety of functions over a somewhat greater range of arguments but they are much briefer. Interpolation is consequently often a major difficulty in these tables, whereas interpolation is relatively easy in the present tables. This can be illustrated roughly by a comparison of the numbers of pages: 230 in Kotani, 305 in Preuss (2 vols.), and 1224 in the volume under review.

A number of research groups interested in quantum chemistry have access to their own high-speed computers, and the present tables will probably not be used much by them except to check out new codes, but a large number of workers whose computational aids are limited, for the most part, to desk calculators will be very happy to see this book. The authors and publisher are to be commended for their efforts in making these tables available.

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1. M. KOTANI, A. AMEMIYA, E. ISHIGURO & T. KIMURA, *Table of Molecular Integrals*, Maruzen Co., Ltd., Tokyo, 1955.
2. H. PREUSS, *Integraltafeln zur Quantenchemie*, Springer-Verlag, Berlin, vol. I, 1956, vol. II, 1957.

37[X].—L. IVAN EPSTEIN, *Nomography*, Interscience Publishers, Inc., New York, 1958, x + 134 p., 24 cm. Price $4.50.

The contents are: Ch. I, Determinants; Ch. II, Nomograms; Ch. III, Projective Transformations; Ch. IV, Matrix Multiplication; Ch. V, More Than Three Variables; Ch. VI, Empirical Nomography; Ch. VII, Kellogg's Method; Ch. VIII, Nonprojective Transformations; Bibliography and Index.

According to the preface, Chs. I, II, III, V, and VI form an elementary text, for which only a knowledge of the elements of analytical geometry is required. Omission of Ch. IV, it is said, will not cause a loss of continuity. For Chs. VII and VIII, a knowledge of calculus is expected. The book attempts, according to the author, to fulfill a need for a book which "combines the discussion and methods of construction with a thorough presentation of the underlying theory", as well as to "make the presentation mathematically rigorous insofar as this could be done on a relatively elementary level".

The book has much to recommend it. The figures are good and the prose in most instances is lucid. The idea of presenting the notion of projective transformations from the "geometrical" point of view first seems to have merit, and useful descriptive terms such as shear and stretch assist in this endeavor. The entire approach to the subject is from the "determinant" point of view, which is desirable. Furthermore, the necessary properties of determinants and matrices are included (Chs. I and IV) for the benefit of the reader unfamiliar with them. In addition, the author does not hesitate to make the natural extension of using the word nomograph as a verb.

On the other hand, there are errors in the book. These are mathematical, pedagogical, grammatical or typographical.

Among the mathematical errors may be mentioned the definition of linearly related functions (p. 93), and some of its consequences. The statement of Theorem

7.1 ("A set of functions of a single variable are linearly related if and only if their wronskian vanishes") shows that the author intended to define what are usually called linearly dependent functions, since Theorem 7.1 would be true if the word "related" were replaced by "dependent," but is *not* true for the definition given, namely, "we shall say that four functions $f_1(x), f_2(x), f_3(x), f_4(x)$ are linearly related if there exist four *nonzero* [emphasis supplied by the reviewer] quantities c_1, c_2, c_3, c_4 independent of x such that $c_1f_1 + c_2f_2 + c_3f_3 + c_4f_4 = 0$". The word *nonzero* should, of course, be replaced by the phrase *not all zero*. Incidentally, "Wronskian" is usually capitalized.

Neither in Ch. VIII nor elsewhere was the reviewer able to discover the author's definition of the phrase "nonprojective transformation". Yet Ch. VIII, based largely on a paper by Gronwall (ref. 5 in the bibliography), has the phrase as its title. The reviewer was also unable to discover the phrase anywhere in Gronwall's paper, from which Theorem 8.1 (concerned with nonprojective transformation) is stated to have been taken.

The reviewer suggests that a worked example involving at least one scale with a curved support would have been appropriate for Ch. II. On p. 23, for equation (2.25) it would have been better to define the units in which the various physical quantities are measured. On p. 83, "antiparallel" should be defined. The reviewer doubts that a reader with merely a knowledge of the calculus would be able (as seems to be implied in the preface) to understand adequately the material of Chapters VII and VIII, even if these chapters were error-free. It is also suggested that too much may have been left as an exercise for the reader in several places.

In conclusion, the reviewer suggests that the unsophisticated reader follow the suggestion made in the preface that Chs. I, II, III, V, and VI form an elementary text. Ch. IV seems to be relatively elementary also. The sophisticated reader should go to the original papers of Kellogg and Gronwall, on which Chapters VII and VIII are respectively based.

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38[X].—ALEXANDER S. LEVENS, *Nomography*, 2nd edition, John Wiley & Sons, New York, 1959, vii + 296 p., 25 cm. Price $8.50.

Professor Levens' *Nomography* is judged by this reviewer to be an excellent elementary textbook. It reflects the author's mastery of pedagogic technique. This work should lead the student to more than the acquisition of a theoretical knowledge of nomography—it should enable him to become a skilled and experienced nomographer. To develop the requisite skill, the text abounds with realistic problems, taken in many cases from practical engineering or physical situations.

The geometric approach in this book is both simple and direct. Perhaps some will find the text overly simple in parts, with too many explicit steps. However, it is likely that most students will appreciate its clarity and ease of reading.

After a brief introduction and a careful discussion of functional scales, the author divides the study into various nomogram types, which are taken up serially. The geometry of each general type is thoroughly discussed and fully illustrated with a detailed application. The geometric approach in this text is considered to

be a far better introduction to nomography than an analytical approach by the theory of determinants. Basic analytic theory is not neglected, however, and a chapter on determinants is included near the end.

The second edition is considered a definite improvement over the first; it has a more pleasing format and type arrangement and an over-all attractiveness that gives it more pedagogic appeal. (One minor defect: chapter numbers on the top of each page of the first edition were unaccountably omitted in the second edition.)

Besides numerous text revisions, important new material appears in the second edition. Most significant are: (a) the expansion of the chapter on determinants, (b) the addition of a chapter on projective transformations, and (c) the addition of a chapter indicating the relationship between concurrency and alignment nomograms (with applications to experimental data, including a description of the rectification of experimental curves).

An appendix supplies an assortment of nomographic solutions to different problems taken from various technical fields, and offers fine illustration of available techniques.

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39[X, Z].—LYLE R. LANGDON, *Approximating Functions for Digital Computers.* Reprinted from *Industrial Mathematics* v. 6, 1955, p. 79–100.

This article is concerned with several methods for determining approximations to functions of one real variable. The methods mentioned include the use of Padé approximants, a modification of the Taylor expansion said to be due to Obrechkoff, the use of Chebyshev polynomials to economize truncated power series in the sense of Lanczos, the use of a rational interpolant which collocates $f(x)$ at five points, as well as some special devices based on a study of the particular function to be approximated.

The following approximations are given:

Function	Nature of approximation	Range of x	Stated upper bound for error
$\sin x$	Rational	$-\pi \leqq x \leqq \pi$	6×10^{-9}
$\cos x$	Rational	$-\pi \leqq x \leqq \pi$	1×10^{-9}
$\tan x$	Rational	$-\pi/4 \leqq x \leqq \pi/4$	7×10^{-8}
e^x	Rational	$-\pi \leqq x \leqq \pi$	1 unit in 9th significant digit
$\sqrt{x}$	Rational	$0.1 \leqq x \leqq 10.0$	Something in 5th significant digit
$\sin \frac{\pi x}{2}$	Polynomial	$-1 \leqq x \leqq 1$	4×10^{-9}
$\cos \frac{\pi x}{2}$	Polynomial	$-1 \leqq x \leqq 1$	7×10^{-9}

Function	Nature of approximation	Range of x	Stated upper bound for error
$\log_e x$	Polynomial in $y = \frac{x-a}{x+a}$, a being a suitably chosen constant	$0.10 \leqq x \leqq 100.0$ (for various choices of a)	3×10^{-9}
$\tan^{-1} y$	Polynomial	$-(\sqrt{2}-1) \leqq y \leqq (\sqrt{2}-1)$	4×10^{-9}
(used to represent $\tan^{-1} x$ on $0 \leqq x \leqq 1$ and $1 \leqq x \leqq \infty$ by expressing y as the ratio of two appropriately chosen linear functions of x)			
$\sin^{-1} x$	Polynomial	$-\frac{1}{2} \leqq x \leqq \frac{1}{2}$	3×10^{-9}
$\sin^{-1} x$	$\frac{\pi}{2} - \sqrt{1-x}\cdot$polyn. in x	$0.966 \leqq x \leqq 1.0$	3×10^{-9}
$H(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$	Polynomial	$0 \leqq x \leqq 1$	6×10^{-10}
	Polynomial	$1 \leqq x \leqq 2$	2×10^{-10}
	Polynomial	$2 \leqq x \leqq 3$	8×10^{-9}
	Polynomial	$3 \leqq x \leqq 4$	7×10^{-9}
$\cosh x \cos x$	Polynomial	$-\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}$	1×10^{-10}
$\cosh x \sin x + \sinh x \cos x$	Polynomial	$-\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}$	2×10^{-10}
$\sinh x \sin x$	Polynomial	$-\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}$	1×10^{-8}
$\cosh x \sin x - \sinh x \cos x$	Polynomial	$-\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}$	3×10^{-9}

(Linear combinations of the last four functions give such functions as

$$e^{\pi x/2} \sin \pi x/2, \int e^x \sin x dx, \text{ etc.})$$

In addition, approximations of certain functions occurring in *Gas Tables* by J. H. Keenan & Joseph Kaye, Wiley & Sons, 1945, are given.

The paper has been reprinted in both an "uncorrected" and a "corrected" version. The following typographical errors were found to remain in the "corrected" version:

Page		
87	line 4 *fb*	Decimal point is missing from .00832 86830x^5, and x^7 is missing from 0.00019 22123x^7.
89	lines 15, 17, 18, 23, 25	Approximate value of $\sqrt{2} - 1$ should end in 4 instead of 3.
90	lines 8, 10	
93–4	Eqs. (15), (16), (17)	Prefix a_0 to the given integral.
93	Eq. (15)	Summation should be $\sum_1^9 a_n(2x-3)^n$.
99	Range of h for Eq. (26)	Should read $395.74189\ 76 < h \leqq 513.4$.

The reviewer compared Langdon's results with those of Cecil Hastings, Jr., *Approximations for Digital Computers*, where possible. Although both authors considered approximations for $\tan^{-1} x$, $\sin(\pi/2)x$, $\sin^{-1} x$, and $H(x)$, the only one of these in which the ranges are the same ($-1 \leqq x \leqq 1$) is that for $\sin(\pi/2)x$. Here both authors use a five-term polynomial approximant; Langdon claims an error bound of 4×10^{-9}, compared to Hastings' claim of 5×10^{-9}. It is of interest to note that C. W. Clenshaw approximated the same function on the same range with a six-term polynomial, the stated error bound being 3×10^{-9} (*MTAC*, v. 1954, p. 143).

The reviewer made some spot checks of the approximations given and found no evidence that the error bounds or coefficients are incorrect, with the exception that the value of $H(2.5)$ was found to be 0.99959 30388, as compared with the value 0.99959 30480 given in the NBS *Tables of Probability Functions*, v. 1. Here the error in the value obtained from Langdon's approximation exceeds the stated bound of $8 \cdot 10^{-9}$.

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40[Z].—H. FREUDENTHAL, "Logique Mathématique Appliquée," *Collection de Logique Mathématique*, Série A, XIV, Gauthier-Villars, Paris, 1958, 57 p. Price $3.00.

This pamphlet is a prize-winning essay in a 1953 contest of the Institute for the Unity of Sciences on the general subject, Mathematical Logic as a Tool of Analysis—its uses and achievements in the sciences and philosophy. For the purposes of the present publication the most interesting portion is the second section entitled "Le calcul des propositions, les réseaux électriques et les machines à calculer." This gives a brief description of various electronic and electro-mechanical realizations of the propositional calculus, introduces the temporal problem, and suggests research in a temporally affected propositional calculus. Unfortunately, the span of five years between initial composition and publication means that no knowledge of the recent work on such temporal structures as the neuron model is demonstrated. The author's point that mathematical logicians should enter the field rather than surrender it to computer engineers retains its initial force.

Briefly, the introductory section comments on what the author considers as an unfortunate tendency to abstract, non-realizable research by logicians. (He concedes the right of number theorists to an ivory tower but thinks that logic must be primarily viewed as a part of applied mathematics.) The third section reviews certain clarifications introduced in philosophy and the foundations of science by the influence of mathematical logic. The fourth suggests an operational definition of implication in terms of an idea of "complete implication" which requires that all pairs (p, q) such that $p \rightarrow q$ be meaningful in a particular case before using the term "implication." The final portion suggests problems involving transitivity in modal forms.

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41[Z].—JOACHIM JEENEL, *Programming for Digital Computers*, McGraw-Hill Book Co., Inc., New York, 1959, viii + 517 p., 24 cm. Price $12.00.

This book is designed for the reader who wishes to become acquainted with stored-program calculators. It is not necessary for the reader to have had previous programming experience; however, a background in college mathematics is desirable. The principles and procedures presented apply mainly to the solution of scientific and technical problems. Some discussion of commercial data processing techniques is also given. The book covers recent developments in the field of programming. The approach is general, in that the procedures which are presented can apply to any stored-program calculator.

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TABLE ERRATA

274.—L. ARNDT, "Recherches sur le calcul des forces perturbatrices dans la théorie des perturbations séculaires," *Bulletin de la Société des Sciences Naturelles de Neuchatel*, v. 24, 1895–1896, p. 3–44.

On page 40, in the table of the hypergeometric function $F(\frac{1}{12}, \frac{5}{12}; 1, x)$ the following corrections should be made: corresponding to $x = 0.650$, for 1.031 8686, read 1.031 9686; and corresponding to $x = 0.651(.001)0.680$ the third decimal places of all tabular values should be increased by a unit.

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275.—E. CAHEN, *Théorie des Nombres*, v. 2, Hermann & Cie, Paris, 1924.

On p. 55, in column 4 of the table of primitive roots of primes, the arguments bracketing 883 should read 881 and 887, respectively; and on p. 56, in column 4, the argument following 2693 should read 2699.

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276.—P. L. CHEBYSHEV, *Teoria delle Congruenze*, Italian translation by I. Massarini, Ermanno Loescher & Co., Rome, 1895.

J. P. KULIK, "Über die Tafel primitiver Wurzeln," *Journal für die reine und angewandte Mathematik*, v. 45, 1853, p. 55–81.

The following corrections should be made in the tables of primitive roots of primes appearing in the Chebyshev volume.

p	for	read	page
19	12	13	248
59	57	56	250
79	5	6	252
269	152	153	273
277	34	14	275
311	180	261	280
	218	285	280
349	307	305	286

Corresponding to the last four primes, identical corrections should be made in Kulik's paper on pages 70, 72, 76, and 81, respectively.

These errors and their corrections have been checked by use of the *Canon Arithmeticus* by K. G. J. Jacobi and by computation on an IBM 650.

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277.—J. B. DALE, *Five-Figure Tables of Mathematical Functions comprising Tables of Logarithms, Powers of Numbers, Trigonometric, Elliptic, and other Transcendental Functions*, Second Edition, Edward Arnold & Co., London, 1949.

Page	Function	x	for	read
82	$\ln x$	5.25	1.65832	1.65823
85	e^{-x}	.04	.96080	.96079
		.06	.94177	.94176
87	e^{-x}	4.1	.10657	.01657
87	$\cosh x$	3.3	13.5747	13.5748
90	$\log \sinh x$	2.5	.98177	.78177
	$\log \tanh x$	5.5	$\bar{1}$.99998	$\bar{1}$.99999
103	$\log \Gamma(x)$	1.45	$\bar{1}$.94726	$\bar{1}$.94727
106	$J_0(x)$	0.3	.99763	.97763
	$J_1(x)$	11.1	−.19138	−.19133
111	erf (x)	.18	.20093	.20094
		.66	.64983	.64938

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EDITORIAL NOTE: A detailed description of these tables, including an enumeration of additional errata, appears in *MTAC*, v. 3, 1949, p. 514.

278.—H. B. DWIGHT, "Table of the Bessel functions and derivatives J_2, J_1', J_2', N_2, N_1', N_2'," *Jn. Math. and Phys.*, v. 25, 1946, p. 93–95. H. B. DWIGHT, *Mathematical Tables*, second edition, Dover Publications, New York, 1958.

In the paper cited there appears the erroneous value $-.257665$ for N_2' (7.1). The correct value is $-.274537$. This correction should also be made in the corresponding entry Y_2' (7.1) shown on p. 182 of the book cited above.

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279.—J. A. RILEY & C. BILLINGS, "Gaussian quadrature of some integrals involving Airy functions," *MTAC*, v. 13, 1959, p. 97–101.

The abscissa value which is given as

0.717013550

should be

0.717013474,

the remainder being correct to nine decimals. All the weights are incorrect in at least the last two places; correct nine-decimal values are:

0.114220867
0.113476346

0.111252488
0.107578286
0.102501638
0.096088727
0.088423159
0.079604868
0.069748824
0.058983537
0.047449413
0.035297054
0.022686232
0.009798996.

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280.—HERBERT E. SALZER, "Orthogonal polynomials arising in the numerical evaluation of inverse Laplace transforms," *MTAC*, v. 9, 1955, p. 164–177.

On p. 174 the statement is made that $p_i^{(n)}$, $1/p_i^{(n)}$, and $A_i^{(n)}$, that is, the reciprocals of zeros, zeros, and Christoffel numbers, respectively, of $P_n(x)$, are "correct to only about a unit in the last significant figure that is given." As a result of a more extended computation, the following errors of more than a single unit in the last place should be noted in the table on p. 175–176:

n	i	Function	*For*	*Read*
4	3, 4	$1/p_i^{(n)}$	.18866 3804 ± .06177 4421i	.18866 3804 ± .06177 4417i
6	3, 4	$p_i^{(n)}$	6.47051 3 ∓ 4.90012 1i	6.47051 5 ∓ 4.90012 1i
6	5, 6	$p_i^{(n)}$	7.49064 0 ∓ 1.62149 9i	7.49063 8 ∓ 1.62150 2i
6	3, 4	$1/p_i^{(n)}$	.09821 855 ± .07438 093i	.09821 855 ± .07438 091i
6	5, 6	$1/p_i^{(n)}$	.12752 426 ± .02760 517i	.12752 426 ± .02760 525i
6	5, 6	$A_i^{(n)}$	−185.544 ∓ 917.794i	−185.544 ∓ 917.792i
8	3, 4	$p_i^{(n)}$	7.73869 0 ∓ 8.37088 1i	7.73868 8 ∓ 8.37087 9i
8	5, 6	$p_i^{(n)}$	9.40637 0 ∓ 4.96922 0i	9.40637 0 ∓ 4.96921 7i
8	7, 8	$p_i^{(n)}$	10.16944 4 ∓ 1.64920 3i	10.16944 6 ∓ 1.64920 3i
8	3, 4	$1/p_i^{(n)}$	.05954 718 ± .06441 172i	.05954 718 ± .06441 174i
8	5, 6	$1/p_i^{(n)}$	.08311 501 ± .04390 820i	.08311 501 ± .04390 818i
8	7, 8	$1/p_i^{(n)}$	.09581 390 ± .01553 837i	.09581 388 ± .01553 835i

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281.—G. W. SPENCELEY AND R. M. SPENCELEY, *Smithsonian Elliptic Functions Tables*, Smithsonian Institution, Washington, D. C., 1947.

We recently computed Jacobi's nome q correct to 20S, corresponding to modular angle θ equal to 15° and 45°, respectively. Comparison of these data with corresponding results published to 16S by G. W. and R. M. Spenceley revealed that their approximation to q when $\theta = 15°$ (on pages 59 and 61) is incorrect in

the last place, where the digit 7 should be replaced by 9. Their value of q corresponding to $\theta = 45°$ is correct as shown on pages 179 and 181.

Additional errata in this tabulation of Jacobi's nome have been published previously by Alan Fletcher [1].

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1. *MTAC*, v. 3, 1948–49, p. 280.

282.—G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, second edition, University Press, Cambridge, 1944.

	for	read
p. 313, line 10 from top	$\frac{\nu + m}{2}$	$-\frac{\nu + m}{2}$
p. 340, equation 7	$-\log 2z - \frac{1}{2}\pi i$	$-\log 2z + \frac{1}{2}\pi i$
p. 340, line 2 from top (upper limit of first integral)	$-1 + \infty$	$-1 + \infty i$

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CORRIGENDA

CHIH-BING LING, "Tables of values of 16 integrals of algebraic-hyperbolic type," *MTAC*, Review 173, v. 12, 1958, p. 306.

	for	read
p. 306, line 13	$k!(\sinh 2x \pm \sin 2x)^{-1}$	$[k!(\sinh 2x \pm 2x)]^{-1}$
p. 306, line 20	$(2x)^k l^{-2x}$	$(2x)^k e^{-2x}$.

E. I. PARKER & PAUL J. NIKOLAI, "A search for analogues of the Mathieu groups," *MTAC*, v. 12, 1958, p. 38–43.

	for	read
p. 43, line 1	3799	3779.

L. ROBIN, Table Erratum, 261 *MTAC*, v. 12, 1958, p. 163. The address of the author should read: Recherches Mathématiques, L'Ingénieur en Chef des Télécommunications, 3 avenue de la République, Issy-les-Moulineaux, Seine, France.

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